1. k-Step Ahead Prediction Error Model

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- All delay is factored into k so the constant terms of A, B, C are not zero
- Constant terms of A and C are one (that is, A, C are monic)

2. k-Step Ahead Prediction Error Model

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Want to predict output from $n+k$ onwards or for $n+j$, $j \geq k$

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Noise has past and future terms, to be split

4. Splitting Noise into Past and Future

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All future terms.

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$$\text{III} = \left(f_{j,0} + f_{j,1}z^{-1} + \cdots + f_{j,\mathrm{d}F_j}z^{-\mathrm{d}F_j}\right)\xi(n)/A(z)$$

III term is known from previous measurements

5. Example: Splitting Noise into Past and Future

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Substitute it in the expression for $y(n+j)$, with $j=2$:

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Second term is unknown;

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Second term is unknown; Last term is known.

6. Splitting Noise into Past and Future

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$$\boldsymbol{Ay(n) = Bu(n-k) + C\xi(n)}$$

$$\boldsymbol{y(n+j) = \frac{B}{A}u(n+j-k) + \frac{C}{A}\xi(n+j)}$$

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\end{aligned}
$$

$$
= \frac{B}{A}u(n+j-k) + \frac{F_j}{A}\frac{Ay(n) - Bu(n-k)}{C} + E_j\xi(n+j
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6. Splitting Noise into Past and Future

$$Ay(n) = Bu(n-k) + C\xi(n)$$

$$\begin{aligned} y(n+j) &= \frac{B}{A}u(n+j-k) + \frac{C}{A}\xi(n+j) \\ &= \frac{B}{A}u(n+j-k) + \left[E_j + z^{-j}\frac{F_j}{A}\right]\xi(n+j) \\ &= \frac{B}{A}u(n+j-k) + \frac{F_j}{A}\xi(n) + E_j\xi(n+j) \end{aligned}$$

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\begin{aligned}
&= \frac{B}{A}u(n+j-k) + \frac{F_j}{A}\frac{Ay(n) - Bu(n-k)}{C} + E_j\xi(n+j \\
&= \frac{B}{A}u(n+j-k) - \frac{F_jB}{AC}u(n-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j \\
&= \frac{B}{A}\left[1 - \frac{F_j}{C}z^{-j}\right]u(n+j-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j)
\end{aligned}
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7. Splitting Noise into Past and Future

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From the previous slide,

$$y(n+j) = \frac{B}{A}\left[1 - \frac{F_j}{C}z^{-j}\right]u(n+j-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j)$$

7. Splitting Noise into Past and Future

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7. Splitting Noise into Past and Future

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8. Example: Splitting C/A into E_j and F_j

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$$\frac{1 + 0.5z^{-1}}{1 - 0.6z^{-1} - 0.16z^{-2}} = \frac{C}{A}$$

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$$\begin{array}{r|llll}
 & 1 & +1.1z^{-1} & & \\
\cline{2-3}
1-0.6z^{-1}-0.16z^{-2} & 1 & +0.5z^{-1} & & \\
 & 1 & -0.6z^{-1} & -0.16z^{-2} & \\
\cline{2-3}
 & & +1.1z^{-1} & +0.16z^{-2} & \\
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$$\frac{1+0.5z^{-1}}{1-0.6z^{-1}-0.16z^{-2}} = (1+1.1z^{-1}) + z^{-2}\frac{0.82+0.176z^{-1}}{1-0.6z^{-1}-0.16z^{-2}}$$

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Cross multiply by $\boldsymbol{A}$:

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- $\boldsymbol{E_j}$, $\boldsymbol{F_j}$ are to be calculated.

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An easier method exists to solve

$$\frac{C}{A} = E_j + z^{-j}\frac{F_j}{A}$$

Cross multiply by A:

$$C = AE_j + z^{-j}F_j$$

- C, A, z^{-j} are known
- E_j, F_j are to be calculated.
- Think: How would you solve it?

10. Different Noise and Prediction Models: ARMAX

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ARMAX Model :

$$\boldsymbol{A y(n) = B u(n-k) + C \xi(n)}$$

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$$\begin{aligned} A &\leftarrow A\Delta, \quad B \leftarrow B\Delta \\ C &= E_j A\Delta + z^{-j}F_j \\ \hat{y}(n+j|n) &= \frac{E_j B\Delta}{C}u(n+j-k) + \frac{F_j}{C}y(n) \end{aligned}$$

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Minimum variance control: Minimize the variations in $\boldsymbol{y}$ at $\boldsymbol{k}$:

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$$\boldsymbol{E_k B u(n) + F_k y(n) = 0}$$

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Ay(n) &= Bu(n-k) + C\xi(n) \\
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y(n+j) &= \frac{E_j B}{C} u(n+j-k) + \frac{F_j}{C} y(n) + E_j \xi(n+j)
\end{aligned}
$$

Minimum variance control: Minimize the variations in y at k:

$$
y(n+k) = \frac{E_k B}{C} u(n) + \frac{F_k}{C} y(n) + E_k \xi(n+k)
$$

To minimize $\mathscr{E}\left[y^2(n+k)\right]$. $\xi(n+k)$ is ind. of $u(n)$, $y(n)$

$$
E_k B u(n) + F_k y(n) = 0
$$

$$
u(n) = -\frac{F_k}{E_k B} y(n)
$$

14. Example: Minimum Variance Control

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$$y(n) = \frac{0.5}{1 - 0.5z^{-1}} u(n-1) + \frac{1}{1 - 0.9z^{-1}} \xi(n)$$

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y(n) &= \frac{0.5}{1 - 0.5z^{-1}} u(n-1) + \frac{1}{1 - 0.9z^{-1}} \xi(n) \\
A &= (1 - 0.5z^{-1})(1 - 0.9z^{-1}) \\
&= 1 - 1.4z^{-1} + 0.45z^{-2} \\
B &= 0.5(1 - 0.9z^{-1}) \\
C &= (1 - 0.5z^{-1}) \\
k &= 1
\end{aligned}
$$

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1 - 0.5z^{-1} &= E_1(1 - 1.4z^{-1} + 0.45z^{-2}) + z^{-1}F_1
\end{aligned}$$

Solving,

$$\begin{aligned}
E_1 &= 1 \\
F_1 &= 0.9 - 0.45z^{-1}
\end{aligned}$$

15. Example: Minimum Variance Control

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&= -0.9 \frac{2 - z^{-1}}{1 - 0.9z^{-1}} y(n) \\
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16. Minimum Variance Control for ARIX Model

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Recall

$$\boldsymbol{A y(n) = B u(n-k) + \frac{1}{\Delta}\xi(n)}$$

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Minimum variance control law is obtained by forcing $\boldsymbol{\hat{y}(n+j|n)}$ to be zero:

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For nonminimum phase systems, use an alternate approach