Hands-on Session: Fractional Calculus and Fractional Differential Equations with SCILAB

Scilab and Its Applications to Global Optimization and Fractional Differential Equations

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Fractional Derivative: Successive Differentiation

- nth-order derivative of t^n (n is integer)

$$\frac{d^n}{dt^n} t^n = n!$$

- nth-order derivative of t^m, m, n integers, $m > n$ is

$$\frac{d^n}{dt^n} t^m = \frac{m!}{(m-n)!} t^{m-n}$$

- Use the Euler's Gamma function (Γ) property,

$$n! = \Gamma(n+1)$$

- So we can re-write

$$\frac{d^n}{dt^n} t^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}$$

- Gamma function is defined for positive and negative reals (except for negative integers and zero). So we let m and n to be reals.

Fractional Derivative: Successive Differentiation

- We define fractional derivative of order $\alpha \in \mathbf{R}$, $\alpha \geq 0$, of t^μ, $\mu \in \mathbf{R}$:

$$\frac{d^\alpha}{dt^\alpha} t^\mu := D_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}$$

- Here the condition $\mu > \alpha$ can be relaxed.
- Let's Start!!

1. $$\frac{d^{0.5}}{dt^{0.5}} t^{0.5} = \Gamma(1+0.5) = \Gamma(1.5)$$

2. $$\frac{d^{0.5}}{dt^{0.5}} t = \frac{1}{\Gamma(1.5)} t^{0.5}$$

3. $$\frac{d^{0.5}}{dt^{0.5}}(1) = \frac{1}{\sqrt{\pi t}} \neq 0$$

Exercise 1

1. We wish to find the 0.6th-order derivative of $t^{2.7}$.
2. Take time interval $t = [0 : 0.01 : 5]$.
3. Write a code in Scilab. Store the derivative in the variable `fracd`.
4. Plot:
   1. $t^{2.7}$ and $D^{0.6}(t^{2.7})$ on the same plot.
   2. First derivative and $D^{0.6}$ of $t^{2.7}$ on the same plot.

Mittag-Leffler Function of One Variable

- Important function in fractional calculus.
- Generalization of the exponential (e^t) function.
- It is given as:

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

- Note that $E_1(t) = e^t$.

Exercise 2

- ML function is

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

- Write a code to calculate the ML function.
- Take time interval $t = [0 : 0.01 : 5]$.
- Take upper limit for the summation $M = 100$.
- Take various values of α.
- Plot $E_\alpha(t)$.
- Check for $\alpha = 1$ you get the exponential curve.

Mittag-Leffler Function of Two Variables

- Extension of the ML function of one variable.
- It is given as:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0.$$

- Note that $E_{1,1}(t) = e^t$.

Exercise 3

- ML function of two variables is

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0.$$

- Write a code to calculate the ML2 function.
- Take time interval $t = [0 : 0.01 : 5]$.
- Take upper limit for the summation $M = 100$.
- Try various combinations of α and β.
- Plot $E_{\alpha,\beta}(t)$.
- Check for $\alpha = \beta = 1$ you get the exponential curve.

Exercise 4: Solution of Fractional Oscillator Equation

- Fractional Oscillator is the generalization of **Harmonic Oscillator**.

$$\frac{d^\alpha}{dt^\alpha}y(t) + \omega^{\alpha-\beta}\frac{d^\beta}{dt^\beta}y(t) = 0$$

where $1 \le \alpha \le 2$, and $0 \le \beta \le 1$.

- Initial conditions are $y(0) = 0$, and $\frac{dy}{dt}(0) = 0$.
- Its solution for $\beta = 0$ is:

$$y(t) = tE_{\alpha,2}(-\omega^2 t^\alpha)$$

- Here

$$E_{\alpha,2}(-\omega^2 t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\omega^2 t^\alpha)^k}{\Gamma(\alpha k + 2)}.$$

- Write a code to plot $y(t)$ for $\alpha = 1.85$, and $\omega = 1$.

Exercise 5: Grünwald-Letnikov (GL) Fractional Derivative

- This definition of fractional derivative is based on the generalization of **backward difference rule**.

$$D^{\alpha}_{GL} f(t) := h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^j \binom{\alpha}{j} f(t - jh).$$

- For $f(t) = t$, find the GL fractional derivative at points $t = 0$, $t = 1$, and $t = 10$. Take $h = 0.1$.

- For the same $f(t)$, now evaluate it over the time interval $[0 : h : 5]$.