Range Computation of Polynomial Problems using the Bernstein Form

by

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Outline

- Introduction
- Bernstein form
- Degree elevation of the Bernstein form
- Vertex property
- Subdivision of the Bernstein form

Introduction

- Various qualitative decision issues (*min. cost, max. profit*, etc), from science and engineering can be perceived as optimization problems.

- General optimization problem formulation is

$$\min_{x} \quad f(x)$$

$$\text{s.t.} \quad h_i(x) = 0, \ i = 1, 2, \ldots, m$$
$$g_j(x) \le 0, \ j = 1, 2, \ldots, n$$

- Minimize above problem *globally*

Cont…

- An optimization problem can be reduced to the problem of computing the *sharp range* of polynomials in several variables on box-like domains.
- We solve the problem of finding the sharp range which *encloses* global *minimum* using the Bernstein form of polynomials.
- The Bernstein coefficients of the expansion provide the lower and upper bounds for the range of the polynomial.
- We can perform subdivision of the original box for faster convergence of the range.

Bernstein Form

- Consider the n^{th} degree polynomial ***p*** in a single variable $x \in \mathrm{U} = [0,1]$

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

- Bernstein form of order ***k*** is

$$p(x) = \sum_{j=0}^{k} b_j^k B_j^k(x) \quad , \qquad k \geq n$$

- $B_j^k(x)$ are the Bernstein basis polynomials of degree ***k***

Cont…

- b_j^k are the Bernstein coefficients

$$b_j^k = \sum_{i=0}^{j} a_i \frac{\binom{j}{i}}{\binom{k}{i}}$$

- The unit interval is not really a restriction as any finite interval **X** can be linearly transformed to it.

Properties of Bernstein Coefficients

The range enclosure property of the Bernstein Form

- The Bernstein coefficients provide bounds for range ***p*** of over **U**=[0,1].
- Lemma 1 (**Range lemma**) (Cargo and Shisha, 1966): The range $\overline{p}([0,1])$ is bounded by the Bernstein coefficients as:

$$\overline{p}([0,1]) \subseteq \left[\min_j b_j^k, \max_j b_j^k \right]$$

- Convex hull property:

$$\text{conv}\{(x, p(x))\} \subseteq \text{conv}\{(I / N, b_I(\text{U})) : I \in S_0\}$$

where $S_0 = \{0, n_1\} \times \{0, n_2\} \times \ldots \times \{0, n_l\}$

Cont…

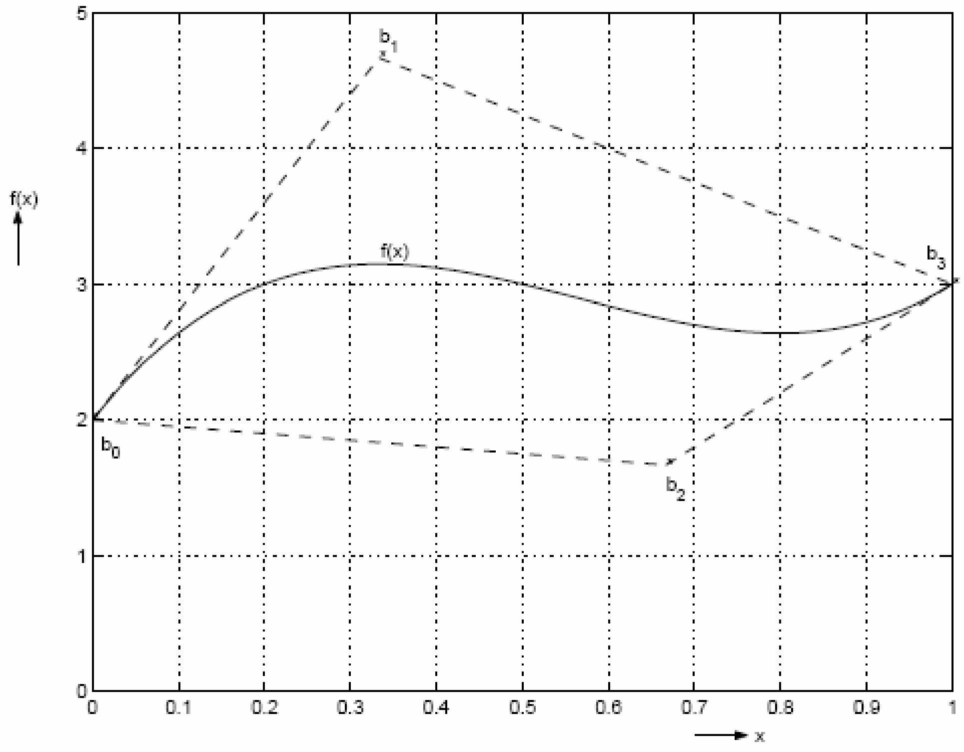

Figure : The polynomial function, its Bernstein coefficients, and the convex hull

Illustration

To illustrate the Bernstein approach for bounding the ranges of polynomials consider the simple polynomial

$$p(x) = x(1-x)$$

whose range $\bar{p}([0,1])$ is $\left[0, \frac{1}{4}\right]$.

- In the Bernstein approach, put polynomial in standard sums of power form

$$p(x) = \sum_{i=0}^{n} a_i x_i$$

where

$$n = 2, a_0 = 0, a_1 = 1, a_2 = -1$$

Cont…

- For $k = 2$ this gives

$$b_0^2 = 0, \quad b_1^2 = \frac{1}{2}, \quad b_2^2 = 0$$

so that

$$\min_j b_j^2 = 0, \quad \max_j b_j^2 = \frac{1}{2}$$

- Range lemma implies

$$\bar{p}\left([0,1]\right) \subseteq \left[0, \frac{1}{2}\right]$$

Vertex Property of Bernstein Form

- Remarkable feature: Bernstein from provides us with a criterion to indicate if calculated estimation is range or not.

- Cargo and Shisha (1966) give such a criterion based on the *vertex* property.

- The *upper bound* or *lower bound* is sharp if and only if $\min b_j^k(\mathrm{U})_{I \in S_0}$ (resp. $\max b_j^k(\mathrm{U})_{I \in S_0}$) is attained at the indices of vertices of Bernstein coefficient array ($B(\mathrm{U})$).

Cont…

Lemma 2 (**Vertex lemma**)

$$\bar{p}\left([0,1]\right) = \left[\min_j b_j^k, \max_j b_j^k\right]$$

if and only if

$$\min_j b_j^k = \min\left\{b_0^k, b_k^k\right\}$$

and

$$\max_j b_j^k = \max\left\{b_0^k, b_k^k\right\}$$

- Vertex lemma also holds for any subinterval of $[0, 1]$.

Illustration

- Consider again the simple polynomial

$$p(x) = x(1-x)$$

whose range $\bar{p}([0,1])$ is $\left[0, \frac{1}{4}\right]$.

- For $k = 4$, Bernstein coefficients are

$$b_0^4 = 0, \quad b_1^4 = \frac{1}{4}, \quad b_2^4 = \frac{1}{3}, \quad b_3^4 = \frac{1}{4}, \quad b_4^4 = 0$$

- Range lemma gives

$$\bar{p}([0,1]) \subseteq \left[0, \frac{1}{3}\right]$$

Cont…

- Check if above enclosure is the range itself or not.

- How? Apply Vertex lemma
  - Minimum Bernstein coefficient is b_0^4 or b_4^4 - occurs at vertices $j \in \{0, 4\}$.
  - Maximum Bernstein coefficient is b_2^4, occurs at $j = 2$ that is not a vertex.
  - Vertex lemma is satisfied for the minimum,
  - Vertex lemma is not satisfied for the maximum - as $\max_j b_j^k \neq \max\{b_0^4, b_4^4\}$.
  - So, by vertex lemma, above enclosure is *not* the range.

Cont…

- Now, we check if any of the range enclosures obtained in previous table for elevated degree of Bernstein form is range or not.
- Table is reproduced below.

Degree k	Range Enclosure	index j for $\min b_j^k$	index j for $\max b_j^k$	Range overestimation
2	$[0, 0.5]$	0	1	0.2500
3	$[0, \frac{1}{3}]$	0	1	0.0833
4	$[0, \frac{1}{3}]$	0	2	0.0833
5	$[0, 0.3]$	0	2	0.0500
6	$[0, 0.3]$	0	3	0.0500
7	$[0, 0.2857]$	0	3	0.0357
10	$[0, 0.2778]$	0	5	0.0278
20	$[0, 0.2632]$	0	10	0.0132
30	$[0, 0.2586]$	0	15	0.0086
100	$[0, 0.2525]$	0	50	0.0025
1000	$[0, 0.2503]$	0	500	0.00025

Cont...

- We find from the table that for any k, the index j for $\max b_j^k$ (in column 4) is not from the vertex set $\{0, k\}$.

- By vertex lemma, *none* of the enclosures in column 2 is the range !

Subdivision of Bernstein Form

- A generally more efficient approach than degree elevation of the Bernstein form is subdivision.

- Let $\mathbf{D} = [\underline{d}, \bar{d}] \subseteq \mathbf{U}$ and assume we have already the Bernstein coefficients on $\mathbf{D}$.

- Suppose $\mathbf{D}$ is bisected to produce two subintervals $\mathbf{D}_A$ and $\mathbf{D}_B$ given by

$$\mathbf{D}_A = [\underline{d}, m(\mathbf{D})]\,; \mathbf{D}_B = [m(\mathbf{D}), \bar{d}]$$

Cont…

- Then, the Bernstein coefficients on the subintervals $\mathbf{D}_A$ and $\mathbf{D}_B$ can be obtained from those on $\mathbf{D}$, by executing the following algorithm.

Subdivision Algorithm

- Inputs: The interval $\mathrm{D} \subseteq \mathrm{U}$ and its Bernstein coefficients (b_j^k).
- Outputs: Subintervals D_A and D_B and their Bernstein coefficients $\tilde{b}_j^k$ and $\hat{b}_j^k$

START

- Bisect **D** to produce the two subintervals D_A and D_B.
- Compute the Bernstein coefficients on subinterval D_A as follows.

(a) $\text{Set}: b_j^k \leftarrow \overline{b}_j^k$, for $j = 0, 1, \ldots, k$

(b) For $i = 1, 2, \ldots, k$ DO

$$\mathrm{b}_j^k = \begin{cases} b_j^{i-1} & \text{for} \quad j < i \\ \frac{1}{2}\left\{ b_{j-1}^{i-1} + b_j^{i-1} \right\} & \text{for} \quad j \geq i \end{cases}$$

Cont…

To obtain the new coefficients apply formula in (b) for $j=0,1,\ldots,k$.

- Find the Bernstein coefficients on subinterval D_A as

$$\tilde{b}_j^{\,k}=b_j^{\,k},\quad \text{for}\quad j=0,1,\ldots,k$$

- Find the Bernstein coefficients on subinterval D_B from intermediate values in above step, as follows.

$$\widehat{b}_j^{\,k}=b_k^{\,j},\quad \text{for}\ \ j=0,1,\ldots,k$$

- Return $\mathrm{D}_A, \mathrm{D}_B$ and the associated Bernstein coefficients $\tilde{b}_j^{\,k}$ and $\widehat{b}_j^{\,k}$.

END

Illustration

- Let us run through Algorithm Subdivision for Example 1.
- For $k = 4$, we have already the Bernstein coefficients $\bar{b}_j^k$ for the interval $\mathbf{D} = [0, 1]$.
- With these as the inputs to Algorithm subdivision, the results at the various steps are

Cont…

- step 1: $\mathbf{D}$ is bisected to produce two subintervals $\mathbf{D}_A = [0, 0.5]$ and $\mathbf{D}_B = [0.5, 1]$.
- step 2: The Bernstein coefficients on subinterval $\mathbf{D}_A$ are computed as follows.
  – step 2a: Set : $b_j^0 \leftarrow \bar{b}_j^4$, for $j = 0, \ldots, 4$,

$$
\begin{aligned}
b_0^0 &= \bar{b}_0^4 = 0; \\
b_1^0 &= \bar{b}_1^4 = \frac{1}{4}; \\
b_2^0 &= \bar{b}_2^4 = \frac{1}{3}; \\
b_3^0 &= \bar{b}_3^4 = \frac{1}{4}; \\
b_4^0 &= \bar{b}_4^4 = 0
\end{aligned}
$$

Cont…

– step 2b:

* for $i = 1$:

$$
\begin{aligned}
b_0^1 &= b_0^0 = 0 \\
b_1^1 &= \frac{1}{2}\left(b_0^0 + b_1^0\right) = \frac{1}{2}\left(0 + \frac{1}{4}\right) = \frac{1}{8} \\
b_2^1 &= \frac{1}{2}\left(b_1^0 + b_2^0\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{3}\right) = \frac{7}{24} \\
b_3^1 &= \frac{1}{2}\left(b_2^0 + b_3^0\right) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{24} \\
b_4^1 &= \frac{1}{2}\left(b_3^0 + b_4^0\right) = \frac{1}{2}\left(\frac{1}{4} + 0\right) = \frac{1}{8}
\end{aligned}
$$

Cont…

$*$ for $i = 2$:

$$
\begin{aligned}
b_0^2 &= b_0^1 = 0 \\
b_1^2 &= b_1^1 = \frac{1}{8} \\
b_2^2 &= \frac{1}{2}\left(b_1^1 + b_2^1\right) = \frac{1}{2}\left(\frac{1}{8} + \frac{7}{24}\right) = \frac{10}{48} \\
b_3^2 &= \frac{1}{2}\left(b_2^1 + b_3^1\right) = \frac{1}{2}\left(\frac{7}{24} + \frac{7}{24}\right) = \frac{7}{24} \\
b_4^2 &= \frac{1}{2}\left(b_3^1 + b_4^1\right) = \frac{1}{2}\left(\frac{7}{24} + \frac{1}{8}\right) = \frac{10}{48}
\end{aligned}
$$

Cont...

$*$ for $i = 3$:

$$b_0^3 = b_0^2 = 0$$

$$b_1^3 = b_1^2 = \frac{1}{8}$$

$$b_2^3 = b_2^2 = \frac{10}{48}$$

$$b_3^3 = \frac{1}{2}\left(b_2^2 + b_3^2\right) = \frac{1}{2}\left(\frac{10}{48} + \frac{7}{24}\right) = \frac{1}{4}$$

$$b_4^3 = \frac{1}{2}\left(b_3^2 + b_4^2\right) = \frac{1}{2}\left(\frac{7}{24} + \frac{10}{48}\right) = \frac{1}{4}$$

Cont…

* for $i = 4$:

$$b_0^4 = b_0^3 = 0$$

$$b_1^4 = b_1^3 = \frac{1}{8}$$

$$b_2^4 = b_2^3 = \frac{10}{48}$$

$$b_3^4 = b_3^3 = \frac{1}{4}$$

$$b_4^4 = \frac{1}{2}(b_3^3 + b_4^3) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}$$

Cont...

– Step 2c: The Bernstein coefficients on the subinterval $\mathbf{D}_A$ are

$$\tilde{b}_0^4 = b_0^4 = 0; \quad \tilde{b}_1^4 = b_1^4 = \frac{1}{8}; \quad \tilde{b}_2^4 = b_2^4 = \frac{10}{48}$$

$$\tilde{b}_3^4 = b_3^4 = \frac{1}{4}; \quad \tilde{b}_4^4 = b_4^4 = \frac{1}{4}$$

- step 3: The Bernstein coefficients on the neighboring subinterval $\mathbf{D}_B$ are

$$\hat{b}_0^4 = b_4^0 = 0; \quad \hat{b}_1^4 = b_4^1 = \frac{1}{8}; \quad \hat{b}_2^4 = b_4^2 = \frac{10}{48};$$

$$\hat{b}_3^4 = b_4^3 = \frac{1}{4}; \quad \hat{b}_4^4 = b_4^4 = \frac{1}{4}$$

Cont...

- step 4: Finally,
  - For subinterval $\mathbf{D}_A$, Bernstein coefficients are

$$\left(0, \frac{1}{8}, \frac{10}{48}, \frac{1}{4}, \frac{1}{4}\right)$$

  - For subinterval $\mathbf{D}_B$, Bernstein coefficients are

$$\left(0, \frac{1}{8}, \frac{10}{48}, \frac{1}{4}, \frac{1}{4}\right)$$

  - It is coincidental here that Bernstein coefficients for both the subintervals are the same.

Cont…

- By range lemma

$$\bar{p}\left(\mathbf{D}_A\right) \subseteq \left[0, \frac{1}{4}\right]$$

$$\bar{p}\left(\mathbf{D}_B\right) \subseteq \left[0, \frac{1}{4}\right]$$

Bernstein Subdivision

- Consider the Bernstein coefficients given a few slides earlier.
- For subinterval $\mathbf{D}_A$,
  - The minimum Bernstein coefficient is $\tilde{b}_0^4$
  - The maximum Bernstein coefficient is $\hat{b}_4^4$.
- Both these occur at the vertices, i.e., for $j \in \{0, 4\}$.
- By the vertex lemma, the range of $\bar{p}\left(\mathbf{D}_A\right)$ is $\left[0, \frac{1}{4}\right]$.

Cont…

- An identical situation holds for other subinterval $\mathbf{D}_B$.
- Thus, we obtain the range $\bar{p}\left([0,1]\right) = \left[0, \frac{1}{4}\right]$.
- In this example, using just *one* subdivision and application of the vertex lemma to the subintervals, we have been able to obtain the range of the given polynomial.
- We are also able to *assert* that obtained enclosure is indeed the range.

Cont…

- It was not possible to get the range through degree elevation, even with Bernstein form of as high a degree as $k = 1000$.
- From Table 1, this high degree Bernstein form still produced an overestimation of about $2.5e - 04$!