Optimization Toolbox using Bernstein Form

by

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Outline

- Introduction
- Bernstein form
- Vertex property
- Subdivision of the Bernstein form
- Cut-off Test
- Global optimization Flow Chart

Introduction

- Optimization: is study of how to find the best (optimum) solution to a problem.
- Objective Function: mathematical representation of a problem.

  e.g. minimum function problem,which is expressed as

$$\min_{x} f(x)$$

where f is the objective function

Cont…

- Global and Local Optima

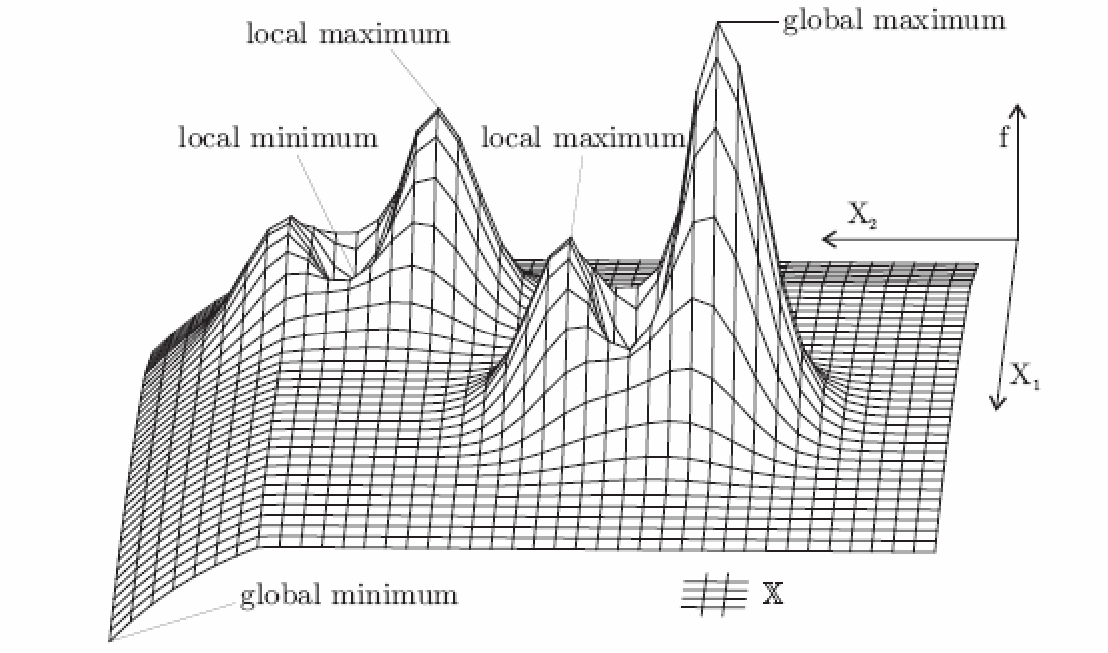
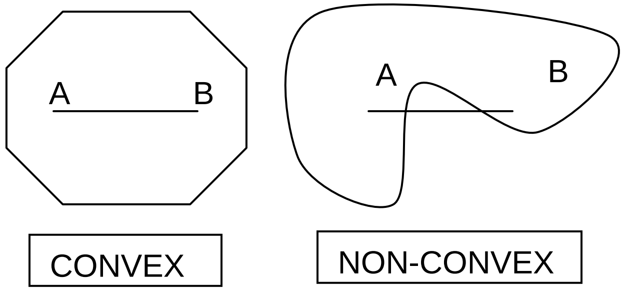


Fig.: Global and local optima of a two-dimensional function

Cont…

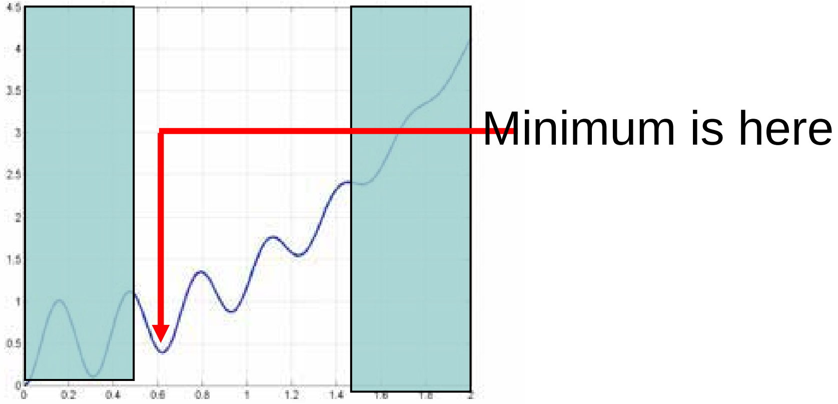
- Convex – Non Convex Function:

  Convex if, every point on the straight line segment that joins them is within the set.



Cont…

- Unconstrained - Constrained Problems



- Univariate/Bivariate/Multivariate Problems
  Single Variable Objective Function = Univariate
  Two Variable Objective Function = Bivariate
  More than one Variable Objective Function = Multivariate

Cont…

- An optimization problem can be reduced to the problem of computing the *sharp range* of polynomials in several variables on box-like domains.
- We can solve the problem of finding the sharp range which encloses global minimum, using the Bernstein form of polynomials.
- The Bernstein coefficients of the expansion provide the lower and upper bounds for the range of the polynomial.
- We can perform subdivision of the original box for faster convergence of the range.

Bernstein Form

- Consider the ***n***th degree polynomial ***p*** in a single variable in power form $x \in \mathrm{U} = [0,1]$

$$p(x) = \sum_{i=0}^{n} a_i x^i \quad a_i \in \mathbb{R} \quad n \in \mathbb{N}$$

,

- Bernstein form of order k is

$$p(x) = \sum_{j=0}^{k} b_j^k B_j^k(x) \quad \forall\ x \in [0,1] \quad k \geq n$$

where $B_j^k(x)$ the Bernstein basis polynomials of degree k and b_j^k Bernstein coefficients

Cont…

- b_j^k are the Bernstein coefficients given by

$$b_j^k = \sum_{i=0}^{j} a_i \frac{\binom{j}{i}}{\binom{k}{i}}$$

- The Bernstein coefficients are collected in an array is called as Bernstein Patch .

Properties of Bernstein Coefficients

- Let $\overline{p}(u)$ denote the range of polynomial p on u. Then, by the range enclosing property of the Bernstein coefficients

$$\overline{p}(u) \subseteq [\min B(u), \max B(u)]$$

which is called 'Range' lemma

- Convex hull property

$$conv\{(x, p(x)) : x \in u\} \subseteq conv\{(I/N, b_I(u)) : I \in S_0\}$$

where

$S_0 = \{0, n_1\} \times \{0, n_2\} \times \ldots \times \{0, n_l\}$ $\quad l \in \mathbb{N}$ and $conv\ p$ denotes convex hull of P

Cont…

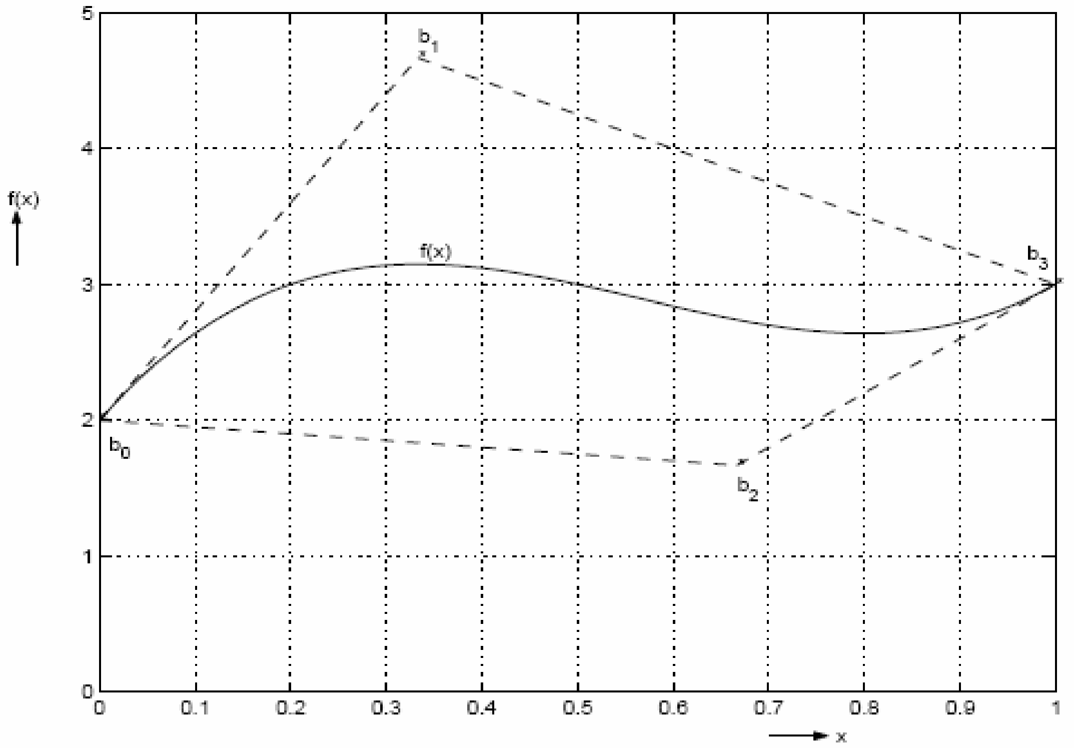


Figure : The polynomial function, its Bernstein coefficients, and the convex hull

Illustration

To illustrate the Bernstein approach for bounding the ranges of polynomials consider the simple polynomial

$$p(x) = x(1-x)$$

whose range $\bar{p}([0,1])$ is $\left[0, \frac{1}{4}\right]$.

- In the Bernstein approach, put polynomial in standard sums of power form

$$p(x) = \sum_{i=0}^{n} a_i x_i$$

where

$$n = 2, a_0 = 0, a_1 = 1, a_2 = -1$$

Cont...

- For $k = 2$ this gives

$$b_0^2 = 0, \quad b_1^2 = \frac{1}{2}, \quad b_2^2 = 0$$

so that

$$\min_j b_j^2 = 0, \quad \max_j b_j^2 = \frac{1}{2}$$

- Range lemma implies

$$\bar{p}\left([0,1]\right) \subseteq \left[0, \frac{1}{2}\right]$$

Assignment 1

- Calculate Bernstein Coefficient for given polynomials

Vertex Property of Bernstein Form

- Remarkable feature: Bernstein Form provides us with a criterion to indicate if calculated estimation is range or not.

- Cargo and Shisha (1966) give such a criterion based on the *vertex* property.

- The *upper bound* or *lower bound* is sharp if and only if $\min b_j^k(\mathrm{U})_{I \in S_0}$ (resp. $\max b_j^k(\mathrm{U})_{I \in S_0}$) is attained at the indices of vertices of Bernstein coefficient array ($B(\mathrm{U})$).

Cont…

Lemma 2 (**Vertex lemma**)

$$\bar{p}\left([0,1]\right) = \left[\min_{j} b_j^k, \max_{j} b_j^k\right]$$

if and only if

$$\min_{j} b_j^k = \min\left\{b_0^k, b_k^k\right\}$$

and

$$\max_{j} b_j^k = \max\left\{b_0^k, b_k^k\right\}$$

- Vertex lemma also holds for any subinterval of $[0,1]$.

Illustration

- Consider again the simple polynomial

$$p\left(x\right)=x\left(1-x\right)$$

whose range $\bar{p}\left([0,1]\right)$ is $\left[0,\frac{1}{4}\right]$.

For k=2, Bernstein Coefficients are

$$b_0^2=0 \qquad b_1^2=\frac{1}{2} or 0.5 \qquad b_2^2=0$$

Range Lemma: $\bar{p}([0,1]) \subseteq \left[0,\frac{1}{2}\right]$

Cont…

- Check if above enclosure is the range itself or not.
- How? Apply Vertex lemma
- Minimum Bernstein coefficient is b_0^2 and b_2^2 - occurs at vertices $j \in \{0, 2\}$

.

- Maximum Bernstein coefficient is b_1^2 - occurs at vertex j =1.

- Vertex lemma is satisfied for the minimum.
- Vertex lemma is not satisfied for the maximum as,

$$\max_j b_j^k \neq \max\{b_0^2, b_2^2\}$$

So, by vertex lemma, above enclosure is not the range.

Cont…

Vertex Test Algorithm:

1. Calculate Bmin from current B (Bernstein patch).

2. Calculate minimum vertices of B. Bvermin.

3. Calculate vertices of B. Bvertex.

4. Check whether Bmin = = Bvermin ?
    1. Then, update stored minimum, zcap = min(Bmin,Bvermin)
       // initial zcap = max(B) taken as worst case.
    2. Check Bmin = = Bvertex. ?
    3. Store current box in solution list, LXsol.

5. If no, go for box subdivision.

Assignment 3

- Write a simple program using Vertex Test Algorithm.

Subdivision of Bernstein Form

- Vertex Condition not satisfied then apply Subdivide test – J. Garloff (1993).

- Let $\mathbf{D} = [\underline{d}, \bar{d}] \subseteq \mathbf{U}$ and assume we have already the Bernstein coefficients on $\mathbf{D}$.

- Suppose D is bisected to produce two sub boxes D_A and D_B

$$\mathbf{D}_A = [\underline{d}, m(\mathbf{D})]\,;\mathbf{D}_B = [m(\mathbf{D}), \bar{d}]$$

Cont…

- Then, the Bernstein coefficients on the sub boxes D_A and D_B can be obtained from those on D by executing the following algorithm.

Subdivision Algorithm

- Inputs: The box $\mathrm{D} \subseteq \mathrm{U}$ and its Bernstein coefficients { $\bar{b}_j^k$ }.
- Outputs: Sub boxes D_A and D_B and their Bernstein coefficients $\tilde{b}_j^k$ and $\hat{b}_j^k$

START

- Bisect **D** to produce the two sub boxes D_A and D_B.
- Compute the Bernstein coefficients on subinterval D_A as follows.

(a) $\text{Set}: b_j^k \leftarrow \bar{b}_j^k$, for $j = 0, 1, \ldots, k$

(b) For $i = 1, 2, \ldots, k$ DO

$$\mathrm{b}_j^k = \begin{cases} b_j^{i-1} & \text{for} \quad j < i \\ \dfrac{1}{2}\left\{ b_{j-1}^{i-1} + b_j^{i-1} \right\} & \text{for} \quad j \geq i \end{cases}$$

Cont…

- To obtain the new coefficients apply formula in (b) for $j=0,1,\ldots,k.$
- Find the Bernstein coefficients on sub box D_A as

$$\tilde{b}_j^{\,k} = b_j^{\,k}, \quad \text{for} \quad j=0,1,\ldots,k$$

- Find the Bernstein coefficients on sub box D_B from intermediate values in above step, as follows.

$$\widehat{b}_j^{\,k} = b_k^{\,j}, \quad \text{for} \;\; j=0,1,\ldots,k$$

- Return $\mathrm{D}_A, \mathrm{D}_B$ and the associated Bernstein coefficients $\tilde{b}_j^{\,k}$ and $\widehat{b}_j^{\,k}$.

END

Illustration

Algorithm Subdivision for Example 1.

For k =4, we have already the Bernstein coefficients $\bar{b}_j^k$ for the interval D= [0,1].

With these as the inputs to Algorithm subdivision, the results are: Bisect D to produce the two sub boxes D_A = [1,0.5] and D_B =[0.5,1].

Cont…

The Bernstein coefficients on the sub box D_A are $\left(0,\frac{1}{8},\frac{10}{48},\frac{1}{4},\frac{1}{4}\right)$

The Bernstein coefficients on the sub box D_B are $\left(0,\frac{1}{8},\frac{10}{48},\frac{1}{4},\frac{1}{4}\right)$

- By range lemma

$$\bar{p}\left(\mathbf{D}_A\right) \subseteq \left[0,\frac{1}{4}\right]$$

$$\bar{p}\left(\mathbf{D}_B\right) \subseteq \left[0,\frac{1}{4}\right]$$

Cont...

In this example, using just *one* subdivision and application of the vertex lemma to the sub boxes we have been able to obtain the range of the given polynomial.

Assignment 4

- Calculate Bernstein Coefficients on subdivided boxes using a scilab code that already implemented in assignment 2.

  (Test above example.)

Cut-off Test

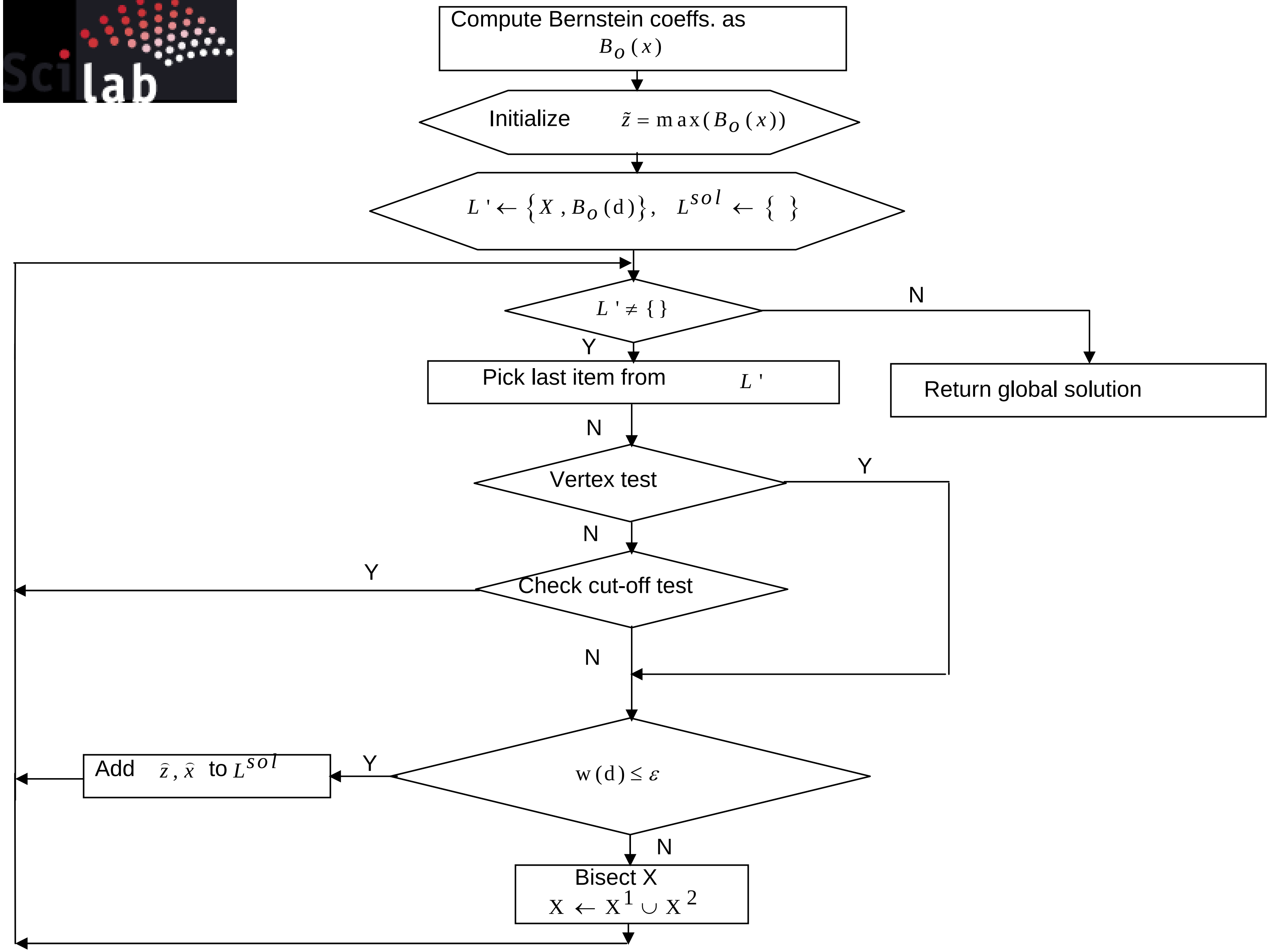
Cut-off Test Algorithm

1. Calculate minimum of B: Bmin
2. Use latest updated zcap
3. Check IF Bmin > zcap, THEN
   delete or discard current box. END
4. Update zcap=min(zcap,Bvermin)

Global Optimization Algorithm

Inputs: Degree N of all variables, polynomial coefficient matrices A for the objective function and specified box X

Output: Global minimum zcap to the specified tolerance $\in$ and all global minimizers $X^{(i)}$


Compute Bernstein coeffs. as $B_o(x)$
Initialize $\tilde{z} = \max(B_o(x))$
$L' \leftarrow \{X, B_o(d)\}$, $L^{sol} \leftarrow \{\ \}$
$L' \neq \{\}$
N
Y
Pick last item from L'
Return global solution
N
Vertex test
Y
N
Y
Check cut-off test
N
Add $\hat{z}, \hat{x}$ to L^{sol}
Y
$w(d) \leq \varepsilon$
N
Bisect X
$X \leftarrow X^1 \cup X^2$

Examples

- **Univariate and Unconstrained Examples with their Global minimum and minimizer**

THANK YOU...