2D minimum-length nozzle contour design with the method of the characteristics in Matlab®

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December, 20th 2020

1 Introduction

A code for the design of the divergent section of a convergent-divergent 2D-nozzle is written using the 2D method of the characteristics for given values of $\gamma = c_p/c_v$ and the exit Mach number M_{exit}. The contour is designed for a constant-property perfect gas by following the procedure introduced by [1]. A planar sonic line at the nozzle throat is assumed.

One major issue is the indexing of the points where the left and right running characteristics intersect. For this sake, a number sequence is used to index the points at the center line.

The code stores the non-dimensional coordinates of the computed contour points in a text file called `nozzle.txt` which can be used to export the nozzle geometry. The code is written for the 2020b release.

2 Physical model

2.1 Governing equations

The present code is designed on the framework adopted by [1]. For the pre-analysis, we assume a **steady, isentropic, compressible, and one-dimensional flow through the divergent section of a minimum-length nozzle**. However, the flow in a 2D nozzle is not one-dimensional and the method of the characteristics takes this aspect into account. The one-dimensional Navier-Stokes equations reduce to the form

$$\frac{\partial}{\partial x}(\rho u) = 0 \tag{1a}$$

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} \tag{1b}$$

$$\frac{\partial H}{\partial x} = 0 \tag{1c}$$

By integrating the above equations one can get the mass conservation equation

$$\dot{m} = \frac{p_o A}{\sqrt{\frac{R}{\mathcal{M}} T_o}} \frac{\sqrt{\gamma} M}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{2}$$

and the energy conservation equation

$$\frac{T_o}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right) \tag{3}$$

(H being the total enthalpy) which, for an isentropic flow, can be written as follows

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{4}$$

Provided that the pressure gradient across the nozzle is strong enough, the Navier-Stokes equations describe an accelerating subsonic ($M < 1$) supersonic ($M > 1$) isentropic (hence shock-free) flow in the convergent and divergent section, respectively. The throat experiences sonic conditions $M = 1$. In this case, the ratio between the exit and throat area is given by

$$\left(\frac{A}{A_t}\right)^2 = \frac{1}{M_{exit}^2} \left[\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M_{exit}^2\right)\right]^{\frac{\gamma+1}{\gamma-1}} \tag{5}$$

In a minimum-length nozzle the expansion takes place at the sharp corner throat, where the expansion waves emanate from. A two-dimensional minimum-length nozzle is symmetric with respect to its central plane ($y/r_{throat} = 0$ in figure 1). Waves generated at on angle intersect the waves generated from the opposite angles and form a complex web of expansion waves in the non-simple region. The increase in Mach number is related to the divergence of the flow through the Prandtl-Meyer function ν

$$\nu(M_2) - \nu(M_1) = \theta \tag{6}$$

$$\nu(M, \gamma) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \arctan \sqrt{\frac{\gamma - 1}{\gamma + 1}(M^2 - 1)} - \arctan \sqrt{M^2 - 1} \tag{7}$$

2.2 The method of the characteristics

The method of the characteristics permits the computation of the contour of a minimum-length nozzle 2D nozzle. In a minimum-length nozzle the flow expands from sonic conditions to the exit conditions immediately after crossing the throat (sonic) section. Hence, the expansion angle of the wall $\theta_{wall,M}$ downstream of the throat is one-half the Prandtl-Meyer function of the exit Mach number M_{exit}:

$$\theta_{wall,M} = \frac{\nu\left(M_{exit}\right)}{2} \tag{8}$$

Equations 5 and 8 can be used to determine the exit area and the initial angle of the nozzle contour for given values of A_t and M_{exit}. However, the nozzle contour and length need to be determined by constructing the characteristic lines. For a symmetric nozzle, the problem reduces to computing the characteristic lines for one-half of the nozzle section (see [1], section 11.7 and figure 11.14). The characteristic lines are the Mach lines, whose inclination is a function of the flow direction θ and Mach angle $\mu = \arcsin\left(1/M\right)$. For a given characteristic

$$\theta \mp \nu\left(M\right) = K_{\pm} \tag{9}$$

$$\left.\frac{dy}{dx}\right|_{char} = \tan\left(\theta \mp \mu\left(M\right)\right) \tag{10}$$

The method exposed in [1] consists in computing a finite number of characteristic curves for finite increments of θ point-by-point by solving the compatibility equations 9. The characteristic curves are approximated with small segments between points. In a first step, the properties defining the characteristics are computed for points of unknown location. The location of each point is subsequently determined by intersecting two given characteristic lines whose slope is a given by average values of θ and μ between the two points.

3 Algorithm

3.1 Points on the first left-running characteristic line

For a given M_{exit}, the total expansion angle downstream of the throat (that is, the angle between the yellow contour and the horizontal in figure 1) is computed from equation 8.

An initial guess value of θ_{guess} is arbitrarily chosen in order to generate the first characteristic (the one impinging in the centerline at point 1, figure 1). The right-running characteristic is symmetric to the left-running characteristic which crosses the centerline at point 1, and the remaining characteristics at points 2, 3, 4, 5, and 6. The remaining points are computed by increasing the amount of θ with a

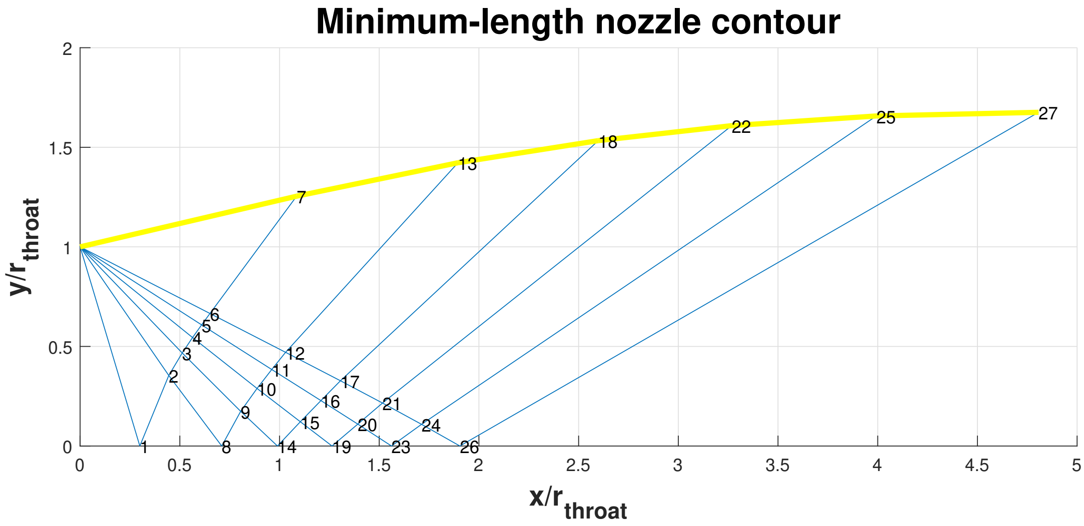

Figure 1: Schematic of the problem with $n = 5$ and $M_{exit} = 2$. Left-running and right-running characteristics intersect at the centerline. Flow properties are computed at each intersection point (numbered). See also figure 11.14 in Anderson, 2003 for a comparison.

fixed step $\Delta\theta$ for an equal number of steps. For instance, in figure 1, a number of $n = 5$ increments (steps) was selected. After point 1, points 2, 3, 4, 5, and 6 are drawn defined. Point 7 is located at the intersection between the left-running characteristic and the contour. Locations are still unknown at this moment.

One needs to determine the values of $K_\pm$, θ, M, and μ at each point.

By applying the Prandtl-Meyer function at the first characteristic line, and by assuming a normal sonic line at the throat, one gets

$$\nu(M_1) - \cancel{\nu(M_t = 1)} = \theta_{guess}$$

M_1 and μ are computed from $\nu(M_1)$, and so are the values of K_- K_+. Values of the right-running K_- are computed in this first step and will be used to compute the properties of the remaining points.

Point 7 draws a simple region over which the properties remain constant. We can therefore assume that properties at point 6 are the same at point 7.

Once the properties have been determined, the coordinates can be obtained by crossing the characteristics. We follow [1] and denote the origin of the expansion fan as point $a(0, 1)$. The coordinates of point 1 will be

$$\begin{cases} x_1 = \dfrac{-y_a}{\tan(\theta_1 - \mu_1)} \\ y_1 = 0 \end{cases}$$

Coordinates of points $i = 2$, 3, 4, 5, and 6:

$$\begin{cases} x_1 = \dfrac{y_{i-1} - 1 - \tan(\theta_{i,i-1} + \mu_{i,i-1})x_{i-1}}{\tan(\theta_i - \mu_i) - \tan(\theta_{i,i-1} + \mu_{i,i-1})} \\ y_i = y_{i-1} + \tan(\theta_{i,i-1} + \mu_{i,i-1})(x_i - x_{i-1}) \end{cases}$$

The approximate location of point 7 is located at the intersection between the slope $\theta_{a7} = \frac{1}{2}(\theta_{max} + \theta_7)$ and the slope $\theta_{6,7} + \mu_{6,7}$ [1, 2].

$$\begin{cases} x_7 = \dfrac{y_6 - 1 - \tan(\theta_{6,7} + \mu_{6,7})x_6}{\tan(\theta_{a,7}) - \tan(\theta_{6,7} + \mu_{6,7})} \\ y_7 = y_6 + \tan(\theta_{6,7} + \mu_{6,7})(x_7 - x_6) \end{cases}$$

3.2 Computation of the remaining points

It is convenient to construct a number sequence to index the remaining centerline points (that is, with $n = 5$, point 8, 14, 19, 23, and 26 in figure 1). Starting from point 8, we have $n-1$ steps in the non-simple region before getting to the contour at point 13. Point 14 is at the centerline, and after $n-2$ steps the characteristic intersects the contour at point 18. This continues till the end. It is convenient to define a sequence of the type:

$$P_0, (n-1), P_{cont,0}, P_1, (n-2), P_{cont,1}, P_2, \ldots$$

Which results in

$$P = P_0 + (n-1) + 2 + (n-2) + 2 + (n-3) + \ldots$$

For the j-th point at the centerline one gets ($P0 = n + 3$)

$$P(j) = n + 3 + jn - \sum_{k=1}^{j} k + 2j$$

$$P(j) = n + 3 + j\left(n + 2 - \frac{j+1}{2}\right) \tag{11}$$

For instance, for $n = 5$ as in the computational scheme in figure 1 one gets $P(j = 0) = 8$, $P(j = 1) = 14$ and so on.

Flow properties are computed starting from the values of K_- and centerline θ. One can compute the value of K_+ for the left-running characteristic curve. The values of M and μ are obtained as before.

The coordinates of the j-th centerline point are defined from $P(j-1)+1$

$$\begin{cases} x_{P(j)} = x_{P(j-1)+1} - \dfrac{y_{P(j-1)+1}}{\tan(\theta_{P(j),P(j-1)+1} - \mu_{P(j),P(j-1)+1})} \\ y_{P(j)} = 0 \end{cases}$$

The coordinates of the points in the non-simple region are given by the intersection of the left-running characteristic from the centerline and the right running characteristic from the throat angle. By denoting points $p = P(j) + i - 1$, $q = P(j) + i$, and $r = P(j-1) + i + 1$, we have

$$\begin{cases} x_q = \dfrac{y_p - y_r - \tan(\theta_{p,q} + \mu_{p,q})x_p + \tan(\theta_{q,r} - \mu_{q,r})x_r}{\tan(\theta_{q,r} - \mu_{q,r}) - \tan(\theta_{p,q} + \mu_{p,q})} \\ y_q = y_p + \tan(\theta_{p,q} + \mu_{p,q})(x_q - x_p) \end{cases}$$

Finally, the contour points are computed from (we denote $p = P(j+1) - 2$ $q = P(j+1) - 1$ $r = P(j) - 1$,)

$$\begin{cases} x_q = \dfrac{y_p - y_q - \tan(\theta_{p,q} + \mu_{p,q})x_p + \tan(\theta_{q,r} - \mu_{q,r})x_r}{\tan(\theta_{q,r}) - \tan(\theta_{p,q} + \mu_{p,q})} \\ y_q = y_p + \tan(\theta_{p,q} + \mu_{p,q})(x_q - x_p) \end{cases}$$

4 Input

The code `nozzle_contour_2D.m` gets the following inputs:

- the exit Mach number M_{exit} `mach_exit`
- the heat capacity ratio γ
- the number of characteristic curves (other than the first one) n `steps`
- the initial *guess* angle necessary to start the computation of the characteristic line θ_{guess} `guess` (should always be small)
- `plot_text` is a logical. If true, prints out a light yellow contour along with the indices of the intersection between the characteristics

References

[1] J.D. Anderson. *Modern Compressible Flow: With Historical Perspective*. Aeronautical and Aerospace Engineering Series. McGraw-Hill Education, 2003.

[2] Zebbiche Toufik and Youbi ZineEddine. Supersonic two-dimensional minimum length nozzle design at high temperature. application for air. *Chinese Journal of Aeronautics*, 20(1):29 – 39, 2007.

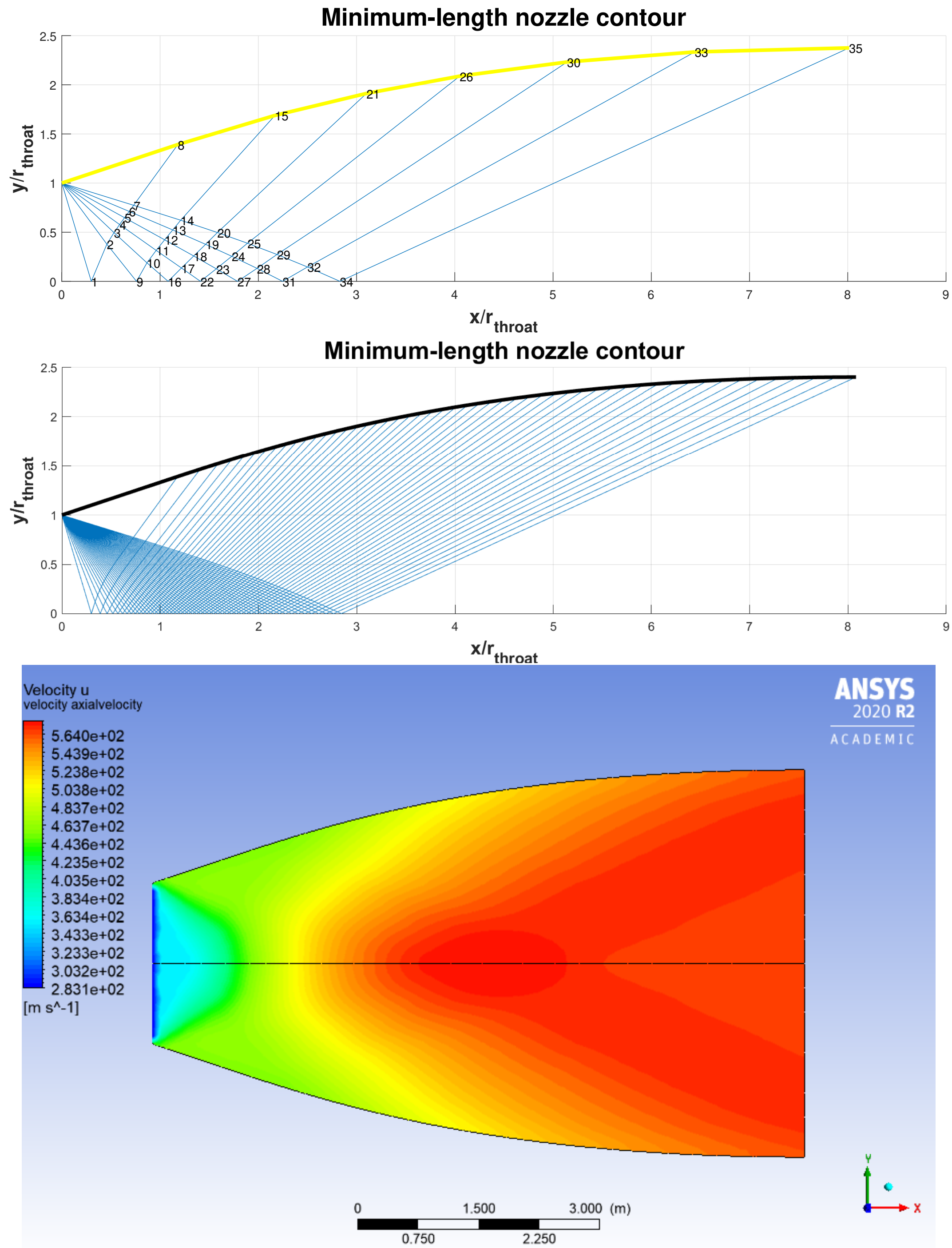

Figure 2: Top: results for $n = 6$, $\gamma = 1.4$ and $M_{exit} = 2.4$, the same as in the example given by Anderson, 2003, section 11.7. Center, same but refined $n = 50$. Bottom: the geometry of the contour is imported to Ansys®Fluent to validate the results (by assuming $p_o = 506625$Pa, $T_o = 300$K in equations 2 and 3 one gets an estimated exit velocity $c = 568\,\mathrm{m\,s^{-1}}$). For a 2D nozzle, following the quasi-1D formulation, for $M_{exit} = 2.4$ we get $A_{exit}/A_t \cong r_{exit}/r_t \cong 2.4031$ from equation 5.

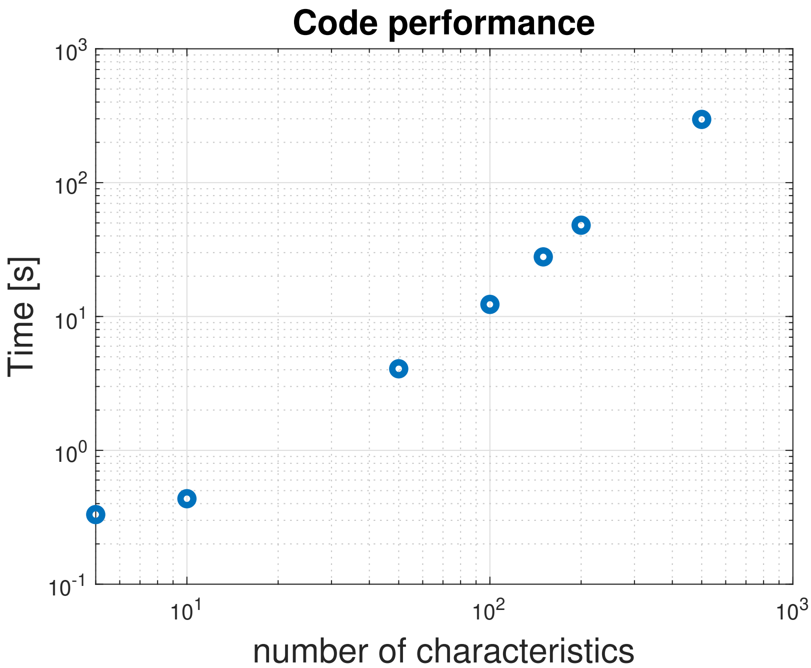

Figure 3: Elapsed computation time compared to the number of the characteristics n requested by input.