Scilab case study project on

Minimum Length Rocket Nozzle Design using Method of Characterises (SCSH25)

Ishar Singh Saini[1]

[1]Department of Aerospace Engineering, Amity University Mumbai, Panvel, Maharashtra

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Abstract

In this study, a methodology based on the Theory of Characteristics is presented for the design of two-dimensional rocket nozzle with minimum-length nozzle configuration. Such a configuration is determined to achieve an optimal Mach number at the exit while ensuring uniform flow in the diverging section. The study has been inspired by Hassan's literature [1] which emphasis on this approach. This approach is then adapted by Fernandes's work [2] who then utilizes a optimization process as a surrogate-based optimization to improve shape and reducing computational power. A numerical approach in Scilab using mathematical, semi-empirical models is developed to solve the governing equations for steady, inviscid, irrotational, and supersonic flow in two dimensions. The analysis assumes that the flow remains consistent in the converging section, allowing for an arbitrary converging profile. Consequently, the primary design focus is placed on the diverging section to optimize performance. The proposed method provides an efficient and precise approach to supersonic nozzle design, ensuring both aerodynamic efficiency and flow uniformity at the exit.

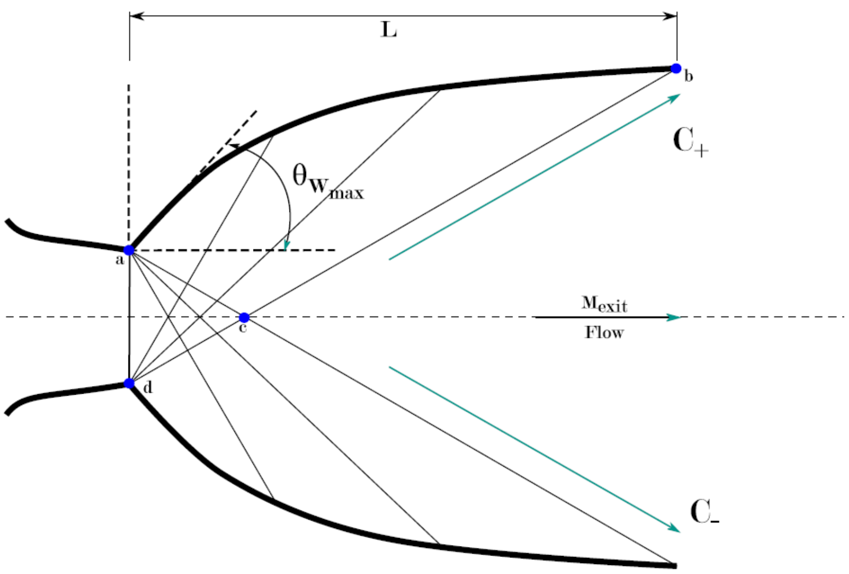

Figure 1. Minimum length nozzle design schematic, adapted from [2]

1. Introduction

A rocket engine nozzle is a crucial component responsible for converting high-pressure, high-temperature combustion gases into a high-velocity jet, generating thrust through Newton's Third Law. It operates on various types of nozzles as shown in Figure 1, but the convergent-divergent (C-D) nozzle, where gases accelerate to sonic speed at the throat and expand supersonically in the diverging section, maximizing exhaust velocity. The nozzle's design, including its shape and expansion ratio, directly influences engine efficiency, specific impulse, and overall performance. Advanced nozzle designs, such as minimum-length nozzles optimized using the Method of Characteristics (MoC), ensure smooth supersonic expansion while minimizing shock losses, making them essential for space propulsion and high-speed atmospheric flight.

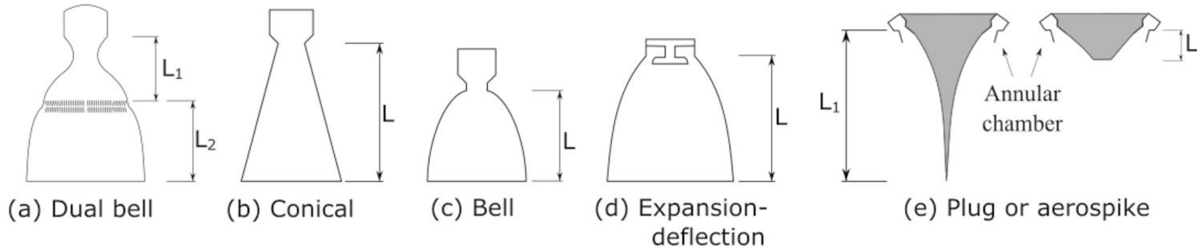

Figure 1. Types of nozzles, adopted from [4]

A convergent-divergent (C-D) nozzle is a fundamental component in high-speed propulsion systems, designed to efficiently accelerate gases from subsonic to supersonic speeds. The throat, the narrowest section of the nozzle, is critical as it governs the transition from subsonic to sonic flow. According to compressible flow theory, when the pressure ratio across the nozzle is sufficient, the flow reaches Mach 1 (sonic condition) at the throat. Beyond this point, the diverging section plays a key role in further accelerating the flow, expanding it to supersonic speeds. The shape of this section is not arbitrary—its contour must be carefully designed to ensure shock-free, isentropic expansion, maximizing efficiency. For a given exit Mach number, the area ratio between the exit and the throat is determined using isentropic flow relations, ensuring that the expansion process is optimized for thrust generation.

The Method of Characteristics (MoC) is a powerful tool used to design the diverging section of a C-D nozzle, ensuring uniform supersonic flow at the exit. In a minimum-length nozzle, expansion occurs instantaneously at the throat via a centered Prandtl-Meyer expansion fan, replacing the need for a gradual expansion contour. The characteristic lines, defined by Mach waves, trace the paths of infinitesimal disturbances, allowing engineers to compute the nozzle wall shape iteratively. By solving the characteristic equations for left- and right-running waves, the method determines the exact contour needed to achieve a shock-free supersonic flow. This is particularly useful in rocket engines, wind tunnels, and scramjet propulsion, where controlling the expansion process ensures maximum thrust and minimal losses due to flow separation or shock formation. This work utilizes this method using Scilab 6.0.0.

2. Problem Statement and Literature Review

Theory of Characteristics is used for the design of two-dimensional rocket nozzle with minimum-length nozzle configuration. Such a configuration is determined to achieve an optimal Mach number at the exit while ensuring uniform flow in the diverging section. The study has been inspired by Hassan's *et al* literature [1] which emphasis on this approach. It builds upon classical compressible flow theories and numerical methods developed in previous studies, including those by Anderson, Rao, and the JANNAF TDK program, which have been widely used for predicting nozzle efficiency. The research focuses on refining the expansion and straightening sections, highlighting the role of Prandtl-Meyer expansion waves in shaping the diverging contour. Unlike traditional bell nozzles, which require gradual flow redirection, this approach condenses the expansion process into a centered expansion fan at the throat, reducing geometric and divergence losses. Theie study also evaluates how grid refinement (increasing the number of characteristic lines) impacts nozzle length and contour smoothness, with results showing stabilization as the grid resolution increases. By comparing different numerical mesh densities, the paper provides insights into optimizing supersonic nozzle design for high-speed propulsion applications, laying a foundation for further computational validation and real-world implementation. Fernandes *et al* [2] presents a low-fidelity rocket nozzle optimization method using the MoC, integrating free-form deformation (FFD) and optimization algorithms to maximize thrust. The study compares MoC with high-fidelity CFD simulations (SU2 framework), using surrogate-based optimization (SBO) to reduce computational costs. While MoC slightly overestimates contour width and thrust, it remains a fast, reliable tool for preliminary nozzle design before high-fidelity CFD. Akhtar's *et al* research [3] highlights the role of grid refinement in smoothing the nozzle contour and stabilizing the solution, making it valuable for high-speed propulsion applications.

Mishra's *et al* study [4] integrates MoC-based calculations with ANSYS validation, comparing theoretical predictions with CFD simulations to assess accuracy. Their results show that MoC slightly overestimates exit Mach numbers due to its ideal assumptions, but it remains a computationally efficient tool for preliminary nozzle design. Tarnacha's *et al* research [5] is a MATLAB-based computational model is used to calculate flow properties such as pressure, density, velocity, and temperature at different nodal points. The nozzle design is further validated through CFD simulations in ANSYS, comparing numerical predictions with realistic flow conditions and their results are same as Mishra's which indicate that while MoC slightly overestimates exit Mach numbers. Kumar's *et al* study [6] shows the same result using the same methodology.

3. Basic Concepts related to the topic

3.1 Governing Equations

For the pre-analysis, we assume a steady, isentropic, compressible, and one-dimensional flow through the divergent section of a minimum-length nozzle. The contour is designed for a constant-property perfect gas by following the procedure introduced by [8]. However, the flow in a 2D nozzle is not one-dimensional and the method of the characteristics takes this aspect into account. The one-dimensional Navier-Stokes equations reduce to the form:

- Continuity Equation

$$\frac{\partial}{\partial x}(\rho u) = 0 \tag{3.1}$$

- Momentum Equation

$$\rho u \left(\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} \tag{3.2}$$

- Energy Equation

$$\frac{\partial H}{\partial x} = 0 \tag{3.3}$$

By integrating the above equations one can get the mass conservation equation and the energy conservation equation:

- Mass Conservation Equation

$$\dot{m} = \left(p_0 \frac{A}{\sqrt{R}}\right)\left(\frac{M}{T_0}\right)\sqrt{\gamma M \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{3.4}$$

- Energy Conservation Equation

$$\frac{T_0}{T} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right) \tag{3.5}$$

(H being the total enthalpy) which, for an isentropic flow, can be written as follows:

- Isentropic Flow Relation

$$\frac{P_0}{P} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{3.6}$$

Provided that the pressure gradient across the nozzle is strong enough, the Navier-Stokes equations describe an accelerating subsonic (M < 1) supersonic (M > 1) isentropic (hence shock-free) flow in the convergent and divergent section, respectively. The throat experiences sonic conditions M = 1. In this case, the ratio between the exit and throat area is given by:

- Area-Mach ratio Relation

$$\left(\frac{A}{A_t}\right)^2 = \left(\frac{1}{M_{exit}^2}\right)\left[\left(\frac{2}{(\gamma+1)}\right)\left(1 + \left(\frac{\gamma-1}{2}\right)M_{exit}^2\right)\right]^{\frac{\gamma+1}{\gamma-1}} \tag{3.7}$$

In a minimum-length nozzle the expansion takes place at the sharp corner throat, where the expansion waves emanate from. A two-dimensional minimum-length nozzle is symmetric with respect to its central plane ($y/r\_throat = 0$ in Figure 1). Waves generated at an angle intersect the waves generated from the opposite angles and form a complex web of expansion waves in the non-simple region. The increase in Mach number is related to the divergence of the flow through the Prandtl-Meyer function ν

- Prandtl-Meyer Function Relationship

$$\nu(M_2) - \nu(M_1) = \theta \tag{3.8}$$

- Prandtl-Meyer Function

$$\nu(M,\gamma) = \sqrt{\frac{(\gamma+1)}{(\gamma-1)}}\tan^{-1}\left(\sqrt{\frac{(\gamma-1)}{(\gamma+1)}}\,(M^2 - 1)\right) - \tan^{-1}\left(\sqrt{M^2 - 1}\right) \tag{3.9}$$

3.2 Method of Characteristics

The method of the characteristics permits the computation of the contour of a minimum-length nozzle 2D nozzle. In a minimum-length nozzle the flow expands from sonic conditions to the exit conditions immediately after crossing the throat (sonic) section. Hence, the expansion angle of the wall $\theta_{wall,M}$ downstream of the throat is one-half the Prandtl-Meyer function of the exit Mach number M_{exit}.

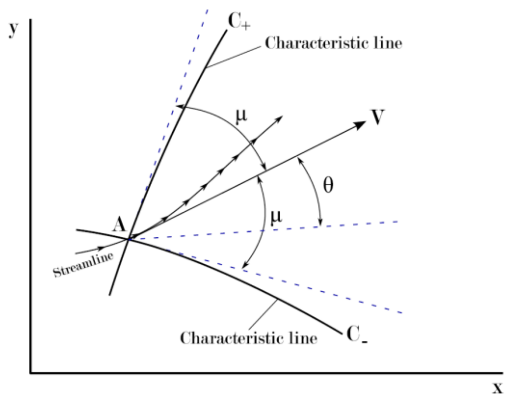

Figure 2. Left- and right- running characteristic lines through point A, adapted from [10]

- Wave Expansion Angle

$$\theta_{wall}, M = \frac{\nu(M_{exit})}{2} \tag{3.10}$$

Equations 3.7 and 3.10 can be used to determine the exit area and the initial angle of the nozzle contour for given values of A_t and M_{exit}. However, the nozzle contour and length need to be determined by constructing the characteristic lines. For a symmetric nozzle, the problem reduces computing the characteristic lines for one-half of the nozzle section [8]. The characteristic lines are the Mach lines, whose inclination is a function of the flow direction θ and Mach angle $\mu = \sin^{-1}(\frac{1}{M})$. For a given characteristic

- Characteristic Equation

$$\theta \mp \nu(M) = K_{\pm} \tag{3.11}$$

- Slope of a Characteristic Line

$$\frac{dy}{dx}|_{char} = \tan\big(\theta \mp \mu(M)\big) \tag{3.12}$$

The method exposed in [8] consists in computing a finite number of characteristic curves for finite increments of θ point-by-point by solving the compatibility equations 3.11 and 3.12. The characteristic curves are approximated with small segments between points. In a first step, the properties defining the characteristics are computed for points of unknown location. The location of each point is subsequently determined by intersecting two given characteristic lines whose slope is a given by average values of θ and μ between the two points.

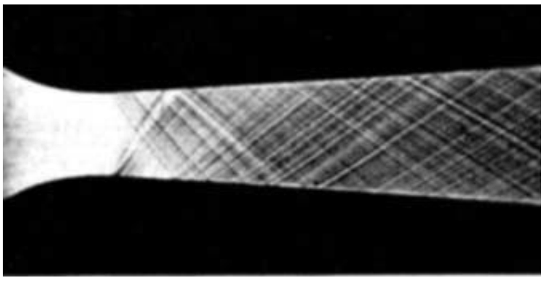

Figure 2. Characterisctic lines visualized in a wind tunnel, adapted from [10]

4. Flow Chart and Algorithm

4.1 Points on the first left – running characteristic line

For a given M_{exit}, the total expansion angle downstream of the throat (that is, the angle between the yellow contour and the horizontal in Figure 3) is computed from equation 3.10

An initial guess value of θ_{guess} is arbitrarily chosen in order to generate the first characteristic (the one impinging in the centerline at point 1, Figure 1). The right- running characteristic is symmetric to the left-running characteristic which crosses the centerline at point 1, and the remaining characteristics at points 2, 3, 4, 5, and 6. The remaining points are computed by increasing the amount of θ with a fixed step $\Delta\theta$ for an equal number of steps. For instance, in Figure 3, a number of n = 5 increments (steps) was selected. After point 1, points 2, 3, 4, 5, and 6 are drawn defined. Point 7 is located at the intersection between the left-running characteristic and the contour. Locations are still unknown at this moment.

One needs to determine the values of $K_{\pm}$, θ, M, and μ at each point.

By applying the Prandtl-Meyer function at the first characteristic line, and by assuming a normal sonic line at the throat, one gets:

$$\nu(M_1) - \cancel{\nu(M_t = 1)} = \theta_{guess}$$

M_1 and μ are computed from $\nu(M_1)$, and so are the values of K_-,K_+. Values of the right-running K_- are computed in this first step and will be used to compute the properties of the remaining points.

Point 7 draws a simple region over which the properties remain constant. We can therefore assume that properties at point 6 are the same at point 7.

Once the properties have been determined, the coordinates can be obtained by crossing the characteristics. We follow [8] and denote the origin of the expansion fan as point $a(0,1)$. The coordinates of point 1 will be

$$\begin{cases} x_1 = -\dfrac{y_a}{\tan(\theta_1 - \mu_1)} \\ y_1 = 0 \end{cases}$$

Coordinates of points i = 2, 3, 4, 5, and 6:

$$\begin{cases} x_i = \dfrac{(y_{i-1} - 1 - \tan(\theta_{i,i-1} + \mu_{i,i-1})\, x_{i-1})}{(\tan(\theta_i - \mu_i) - \tan(\theta_{i,i-1} + \mu_{i,i-1}))} \\ y_i = y_{i-1} + \tan(\theta_{i,i-1} + \mu_{i,i-1}) * (x_i - x_{i-1}) \end{cases}$$

The approximate location of point 7 is located at the intersection between the slopes $\theta_{a7} = \frac{1}{2}(\theta_{max} - \theta_7)$ and the slope $\theta_{6,7} + \mu_{6,7}$ [8, 9]

$$\begin{cases} x_7 = \dfrac{y_6 - 1 - \tan(\theta_{6,7} + \mu_{6,7}) * x_6}{\tan(\theta_{a,7}) - \tan(\theta_{6,7} + \mu_{6,7})} \\ y^7 = y_6 + \tan(\theta_{6,7} + \mu_{6,7}) * (x_7 - x_6) \end{cases}$$

4.2 Computation of the remaining points

It is convenient to construct a number sequence to index the remaining centerline points (that is, with n = 5, point 8, 14, 19, 23, and 26 in Figure 3). Starting from point 8, we have n = 1 steps in the non-simple region before getting to the contour at point 13. Point 14 is at the centerline, and after n - 2 steps the characteristic intersects the contour at point 18. This continues till the end. It is convenient to define a sequence of the type:

$$P_0, (n-1), P_{cont,0}, P_1, (n-2), P_{cont,1}, P_2, \; . \; . \; . \; .$$

Which results in

$$P = P_0 + (n-1) + 2 + (n-2) + 2 + (n-3) + . \; . \; . \; .$$

For the *j*-th point at the centerline one gets ($P_0 = n + 3$)

$$P(j) = n + 3 + jn - \sum_{k=1}^{j} k + 2j$$

$$P(j) = n + 3 + j\left(n + 2 - \tfrac{j+1}{2}\right) \tag{4.1}$$

For instance, for n = 5 as in the computational scheme in Figure 3 one ge ts $P(j=0) = 8$, $P(j=1) = 14$ and so on.

Flow properties are computed starting from the values of K_- and centerline θ. One can compute the value of K_+ for the left-running characteristic curve. The values of M and μ are obtained as before.

The coordinates of the j-th centerline point are defined from $P(j-1)+1$

$$\begin{cases} x_{P(j)} = x_{P(j-1)+1} - \left(\dfrac{y_{P(j-1)+1}}{\tan\left(\theta_{P(j),P(j-1)+1} - \mu_{P(j),P(j-1)+1}\right)}\right) \\ y_{P(j)} = 0 \end{cases}$$

The coordinates of the points in the non-simple region are given by the intersection of the left-running characteristic from the centreline and the right running characteristic from the throat angle. By denoting points $p = P(j) + i - 1, q = P(j) + i$, and $r = P(j-1) + i + 1$, we have:

$$\begin{cases} x_q = \dfrac{y_p - y_r - \tan\left(\theta_{p,q} + \mu_{p,q}\right) x_p + \tan\left(\theta_{q,r} - \mu_{q,r}\right) x_r}{\tan\left(\theta_{q,r} - \mu_{q,r}\right) - \tan\left(\theta_{p,q} + \mu_{p,q}\right)} \\ y_q = y_p + \tan\left(\theta_{p,q} + \mu_{p,q}\right)\left(x_q - x_p\right) \end{cases}$$

Finally, the contour points are computed from (we denote $p = P(j+1) - 2$, $q = P(j+1) - 1 r = P(j) = 1$)

$$\begin{cases} x_q = \dfrac{y_p - y_q - \tan\left(\theta_{p,q} + \mu_{p,q}\right) x_p + \tan\left(\theta_{q,r} - \mu_{q,r}\right) x_r}{\tan\left(\theta_{q,r}\right) - \tan\left(\theta_{p,q} + \mu_{p,q}\right)} \\ y_q = y_p + \tan\left(\theta_{p,q} + \mu_{p,q}\right)\left(x_q - x_p\right) \end{cases}$$

4.3 Flowchart

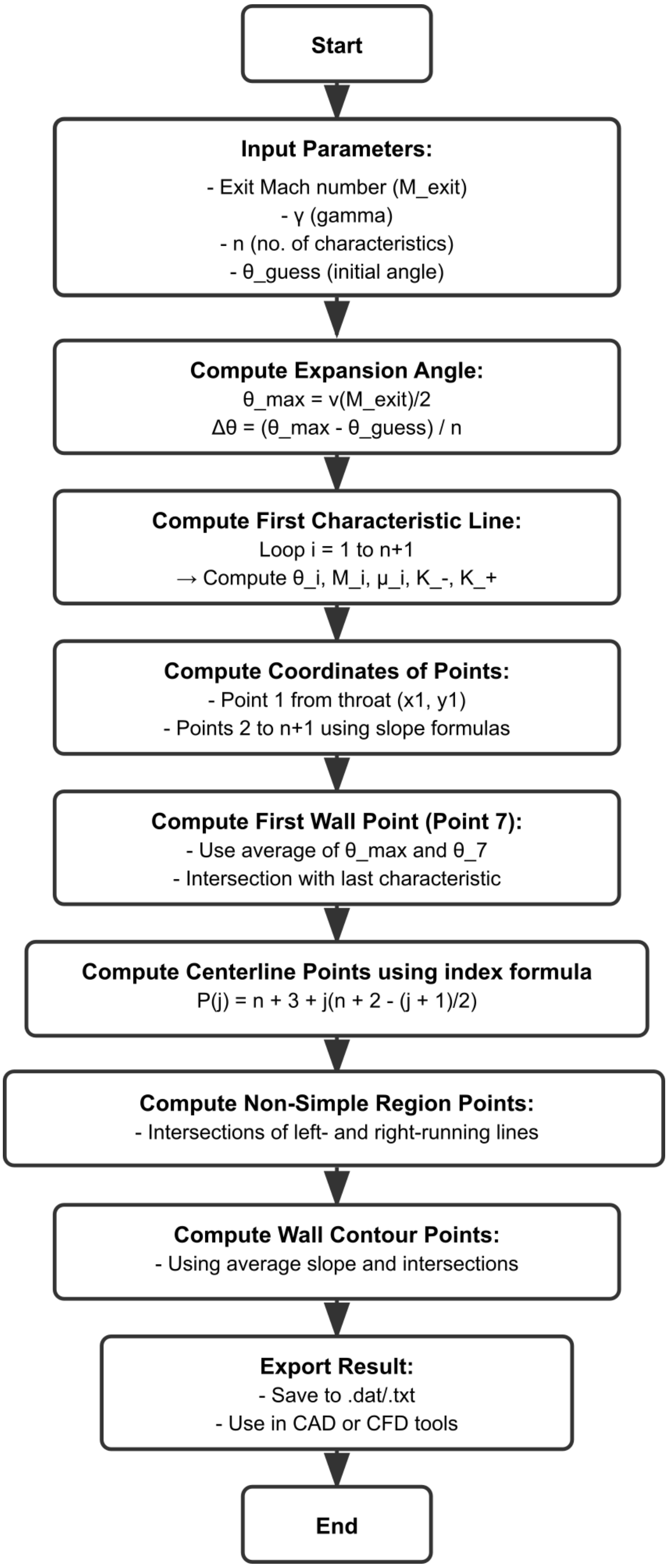

Figure 3. Flowchart explaining the algorithm

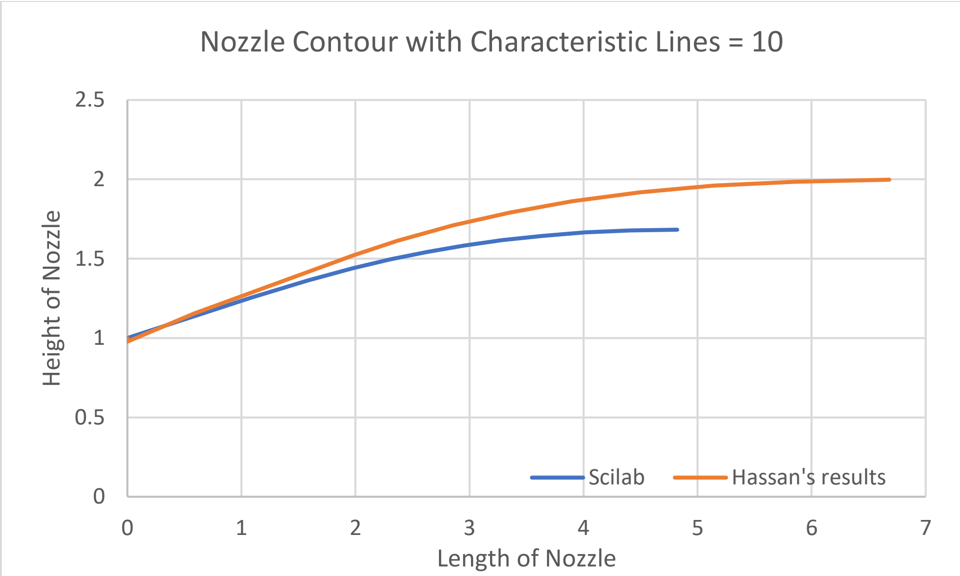

Figure 4 Numerical output for 10 characteristic line

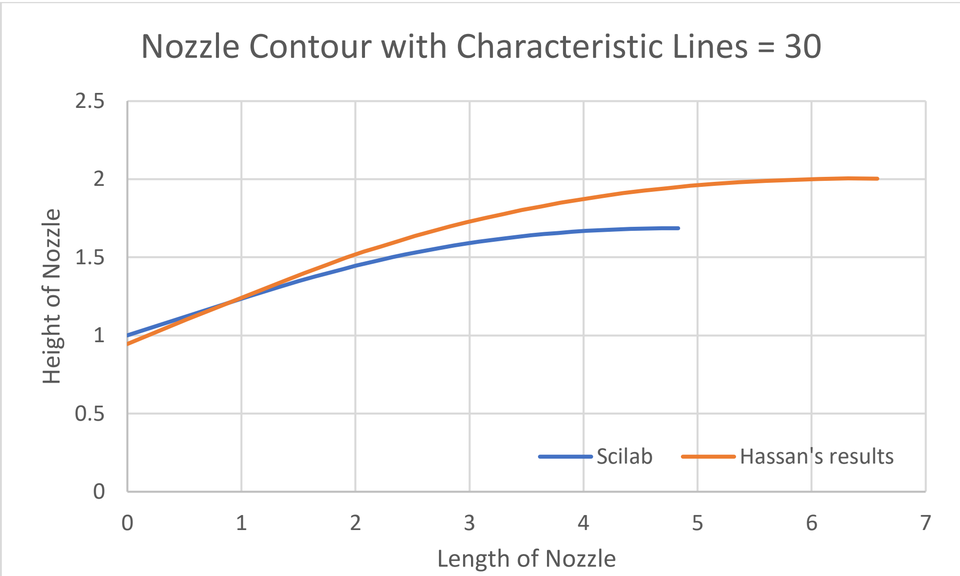

Figure 5 Numerical output for 30 characteristic lines

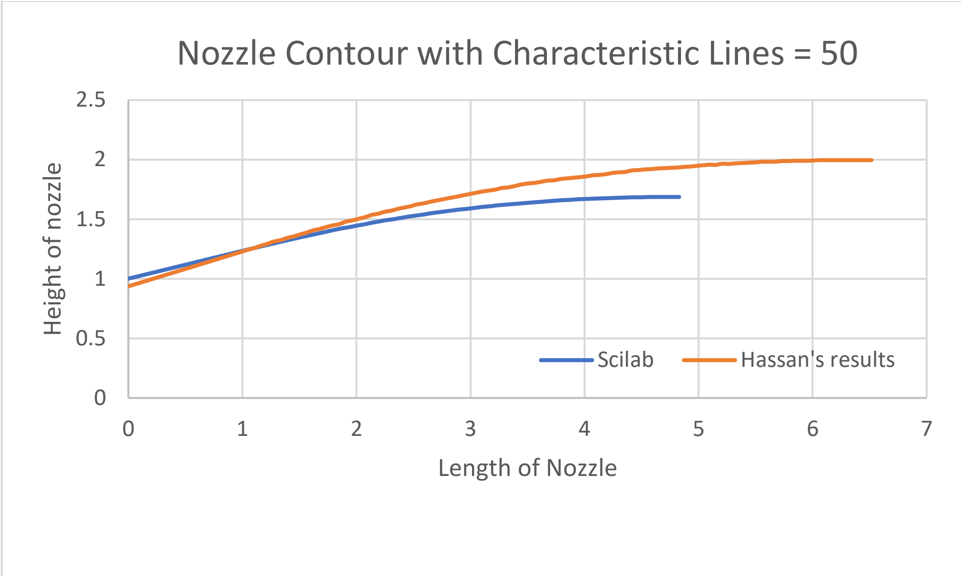

Figure 6 Numerical output for 50 characteristic lines

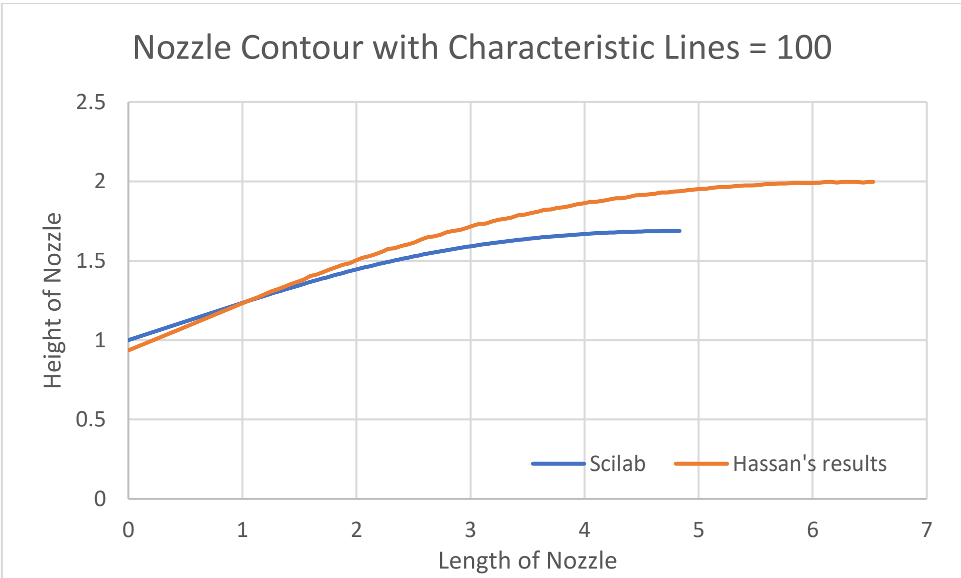

Figure 7 Numerical output for 100 characteristic lines

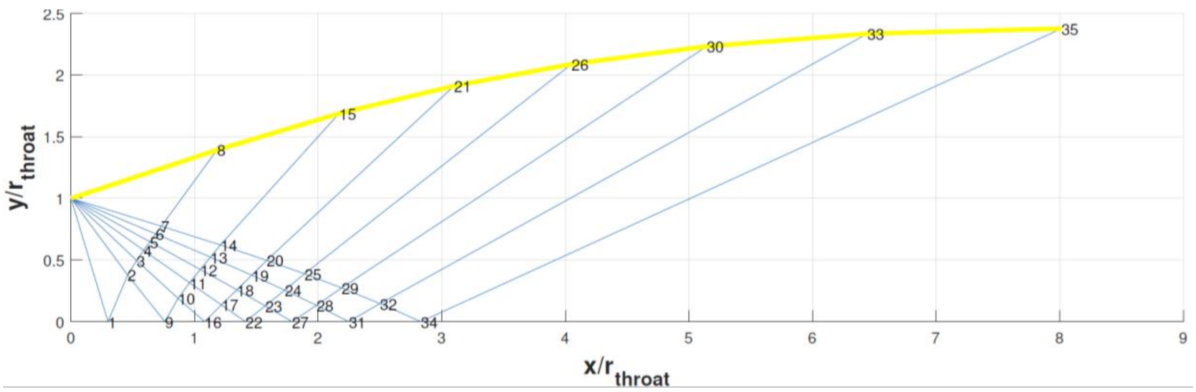

Figure 4. Minimum-length nozzle Contour

As for the inputs required for the code, the exit Mach no. should be defined, the ratio of specific heats (assumed to be 1.4) , an arbitrary guess value for θ_{guess} (0.375 radians as default).

5. Software/Hardware used

The current computional resource utilizes Windows 11 with 6 cores with 16 GB ram. The scilab version used is Scilab-6.0.0 (64-bit).

6. Procedure of Execution

For executing the code, Open Scilab > Open the file "MinRocketNozzle.sci" and input the necessary parameters required to run the code. If not available, just press enter and the code will run on its default values.

7. Results

A comparison of Hassan's results and our results our shown below. Note: the x and y points are non- dimensionized by dividing by the nozzle throat radius. The mesh has been made finer by increasing the number of characteristic lines 10 to 200 in five stages. Throughout the mesh increment process, it has become apparent that finer mesh produces rectified result. The evidence of the result rectification is set by nozzle contour smoothening.

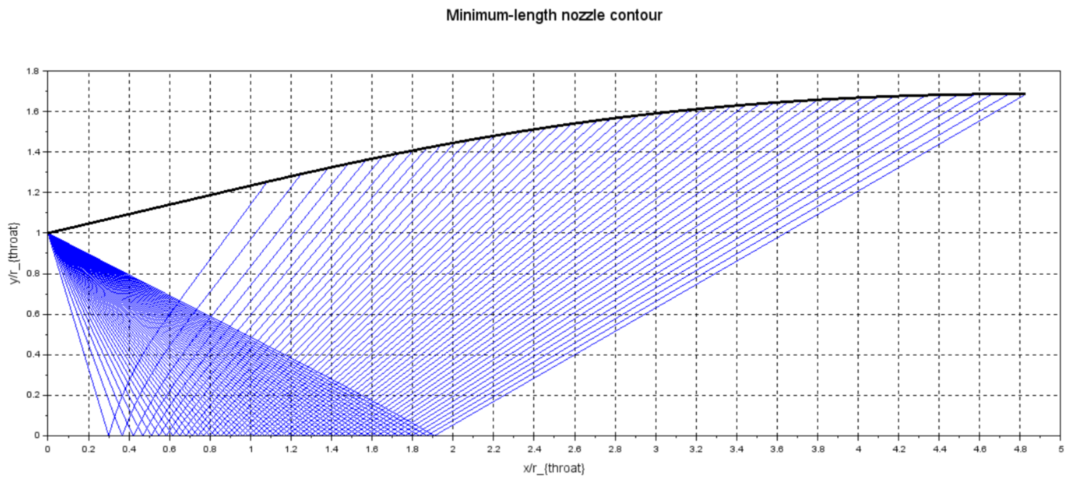

Figure 8 Minimum Length nozzle contour (Scilab result for n = 50)

8. Conclusion

A numerical study was conducted for a computational approach for designing minimum-length supersonic nozzles using the Method of Characteristics (MoC), implemented in Scilab. By assuming isentropic, compressible, and steady flow, the methodology effectively generates a shock-free nozzle contour, ensuring optimal supersonic expansion. The algorithm constructs characteristic lines and computes the flow properties at intersection points, allowing precise determination of the nozzle's diverging section. A literature review of such approach was conducted to confirm that the MoC-based approach provides an efficient, computationally inexpensive tool for preliminary supersonic nozzle design, with the ability to refine accuracy by increasing the number of characteristic lines and which has been used for CFD simulations in other literatures. The study was compared with Hassan's results [1] which showed that the current work not only validates the results but is more optimized to deliver a smaller length nozzle in the same conditions. The simulation program of the practical supersonic nozzle can be developed depending on this design with the considerations of the losses take place in real time.

References

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2. Fernandes, Tiago, Alain Souza, and Frederico Afonso. "A shape design optimization methodology based on the method of characteristics for rocket nozzles." *CEAS Space Journal* 15, no. 6 (2023): 867-879. https://doi.org/10.1007/s12567-023-00511-1
3. Khan, Md Akhtar, Sanjay Kumar Sardiwal, MV Sai Sharath, and D. Harika Chowdary. "Design of a supersonic nozzle using method of characteristics." *International Journal of Engineering and Technology* 2 (2013): 19-24. https://doi.org/10.17577/IJERTV2IS110026