Design of a Supersonic Nozzle using Method of Characteristics

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Design of a Supersonic Nozzle using Method of Characteristics

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Abstract - In this paper, a method based on the theory of characteristics is presented for two-dimensional, supersonic nozzle design. Minimum length of the supersonic nozzle has been calculated for the optimum Mach number at the nozzle exit with uniform flow at the diverging section of the nozzle by developing a MATLAB program. Numerical solution is established for the two dimensional, steady, in viscid, irrotational and supersonic flow. It is rational to assume the flow holds the consistency in the converging section and, thereby, an arbitrary shape is assumed for the converging section of the supersonic nozzle. The design considerations are concentrated at the diverging section.

Keywords: Design, Supersonic, sC-D Nozzle, Minimum Length, Method of Characteristics.

Nomenclature:

M	- Mach number
α	- Local Mach number
υ (M)	-Prandtl Meyer function
υ	- Prandtl Meyer angle
K+K_	-Riemann function
Ө	-Angle made with respect to Streamline coordinate axis
$\theta_{Wmax, ML}$	- Maximum expansion angle for minimum length nozzle

1. Introduction.

The physical conditions of a two-dimensional, steady, isentropic, irrotational flow can be expressed mathematically by the nonlinear differential equation of the velocity potential. The method of characteristics is a mathematical formulation that can be used to find solutions to the aforementioned velocity potential, satisfying given boundary conditions for which the governing partial differential equations (PDEs) become ordinary differential equations (ODEs).

Traditionally, the supersonic nozzle is divided in two parts. The supersonic portion is independent of the upstream conditions of the sonic line. We can study the subsonic portion independently. The latter is used to give a sonic flow at the throat. We design a type of nozzle giving a parallel and uniform flow at the exit section. It is named by Minimum Length Nozzle with centered expansion, which gives the minimal length compared to the other existing types. [4]

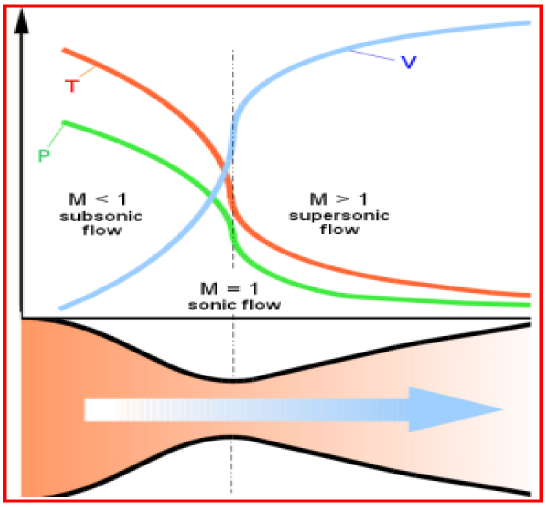

Fig.1 Convergent-Divergent Duct

2. General Theory of Method of Characteristics

Characteristics are 'lines' in a supersonic flow oriented in specific directions along which disturbances (pressure waves) are propagated. The Method of Characteristics (MOC) is a numerical procedure appropriate for solving, among other things, two-dimensional compressible flow problems. By using this technique, flow properties such as direction and velocity, can be calculated at distinct points throughout a flow field. [3]

The three properties of characteristics are as follows:

Property 1: A characteristic in a two-dimensional supersonic flow is a curve or line along which physical disturbances are propagated at the local speed of sound relative to the gas.

Property 2: A characteristic is a curve across which flow properties are continuous, although they may have discontinuous first derivatives, and along which the derivatives are indeterminate.

Property 3: A characteristic is a curve along which the governing partial differential equations(s) may be manipulated into an ordinary differential equation(s).

It is in the region immediately after the sonic throat where the flow is turned away from itself that the air expands into supersonic velocity. This expansion happens rather gradually over the initial expansion region. In the Prandtl-Meyer expansion scenario, it is assumed that the expansion takes place across a centered fan originating from an abrupt corner .This phenomenon is typically modeled as a continuous series of expansion waves, each turning the airflow an infinitesimal amount along with the contour of the channel wall.

These expansion waves can be thought of as the opposite of shock compression waves, which slow airflow. This is governed by the Prandtl-Meyer function:

$$d\theta = \pm\sqrt{M^2-1}\frac{dV}{V} \qquad \text{-----2.1}$$

Where the change in flow angle (relative to its original direction) is represented by $d\theta$. Integrating the above equation to give the following

$$v(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - tan^{-1}\sqrt{M^2-1}$$

----2.2

The parameter ν is known as the Prandtl-Meyer angle.[4]

Method of Characteristics analysis for this project used the following equations; In Method of Characteristics equations the angle of the flow with respect to the horizontal is given the symbol θ. The Mach angle α is defined as $\alpha = \arcsin\left(\frac{1}{M}\right)$. The equations are with the reference figure.2.1 below,

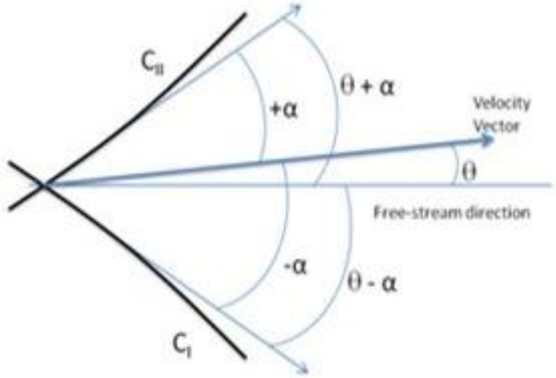

Fig. 2 Schematic diagram of characteristic lines

$$\left(\frac{dy}{dx}\right)_I = \tan(\theta-\alpha)$$

$$\left(\frac{dy}{dx}\right)_{II} = \tan(\theta+\alpha)$$

$\theta + \upsilon(M)$ = constant = K_- (along C_- characteristic) ----2.3

$\theta - \upsilon(M)$= constant = K_+ (along C_+ characteristic) ----2.4

K_- and K_+ are constants along their respective characteristics are and are known as Riemann invariants [3]

Where $\theta = \frac{1}{2}(K_- + K_+)$

and

$\nu = \frac{1}{2}(K_- - K_+)$

Consider the intersection of two characteristic lines A and B at point P,

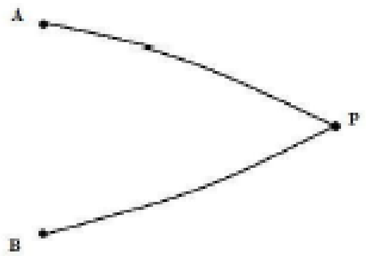

Figure.2.2. Intersection of characteristics lines

then we have,

$$m_I = \tan\left(\frac{(\theta-\alpha)_A+(\theta-\alpha)p}{2}\right)$$

$$m_{II} = \tan\left(\frac{(\theta-\alpha)_B+(\theta-\alpha)p}{2}\right)$$

and

$$y_p = y_A + m_1(x_P - x_A)$$

and

$$y_p = y_B + m_{11}(x_P - x_B)$$

$$x_P = \frac{y_1 - y_B + m_{11}x_B - m_1 x_A}{m_{11} - m_1}$$

Thus, helping in developing the Supersonic nozzle. [3]

3. Nozzle Design

Supersonic nozzles are used in a variety of engineering applications to expand a flow to desired supersonic conditions. Supersonic nozzles can be divided into two different types: gradual-expansion nozzles and minimum-length nozzles (Fig. 2.5). Gradual-expansion nozzles are typically used in applications where maintaining a high-quality flow at the desired exit conditions is of importance (e.g., supersonic wind tunnels). For other types of applications (e.g., rocket nozzles), the large weight and length penalties associated with gradual-expansion nozzles make them unrealistic; therefore minimum-length nozzles, which utilize a sharp corner to provide the initial expansion, are commonly used.

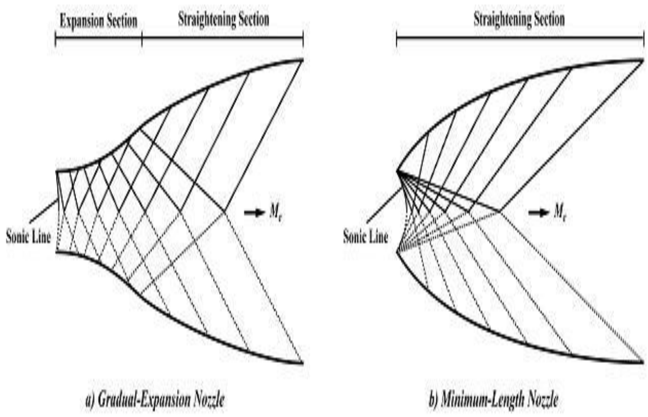

Fig.3. Types of nozzles.

For both gradual-expansion and minimum-length nozzles, the flow can be divided into simple and non-simple regions. A non-simple region is characterized by Mach wave reflections and intersections. In order to meet the requirement of uniform conditions at the nozzle exit, it is desirable to minimize the non-simple region as much as possible. This can be performed by designing the nozzle surface such that Mach waves (e.g., characteristics) are not produced or reflected while the flow is straightened. The Method of Characteristics is therefore applied to allow the design of a supersonic nozzle which meets these requirements. In the present work, design of both gradual-expansion nozzle and minimum-length nozzle is demonstrated. [3]

4. Design of Minimum Length Supersonic Nozzle.

In order to expand an internal steady flow through a duct from subsonic to supersonic speed the duct has to be convergent - divergent in shape. If the nozzle contour is not proper, shock waves may occur inside the duct. The method of characteristics provides a technique for properly designing the contour of a supersonic nozzle for shock free, isentropic flow, taking into account the multidimensional flow inside the duct The purpose of this section is to illustrate such an application.

Rocket nozzles are short in order to minimize weight. Also, in cases where rapid expansions are desirable, such as the non-equilibrium flow in modern gas dynamic lasers, the nozzle length is as short as possible In such minimum- length nozzles, the expansion section is shrunk to a point, and the expansion takes place through a centered Prandtl-Meyer wave emanating from a sharp-corner throat with an angle θ_{W_{max},M_L} as sketched in Fig 3.1 The length of the supersonic nozzle, denoted as L in Fig 3.1 is the minimum value consistent with shock free, isentropic flow. If the contour is made shorter than L, shocks will develop inside the nozzle.

A fluid element moving along a streamline is constantly accelerated while passing through these multiple reflected waves. For the minimum-length nozzle, the expansion contour is a sharp corner at point a. There are no multiple reflections and a fluid element encounters only two systems of waves—the right-running waves emanating from point a and the left-running waves emanating from point d. Let ν_M be the Prandtl-Meyer function associated with the design exit Mach number Hence, along the C_+ characteristic 'cb', $\nu = \nu_M = \nu_c = \nu_b$. Now consider the C_- characteristic through points a and c.

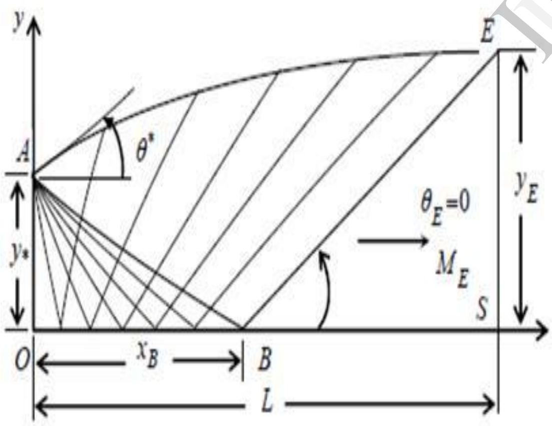

Fig. 3 Schematic of characteristic lines for MLN.

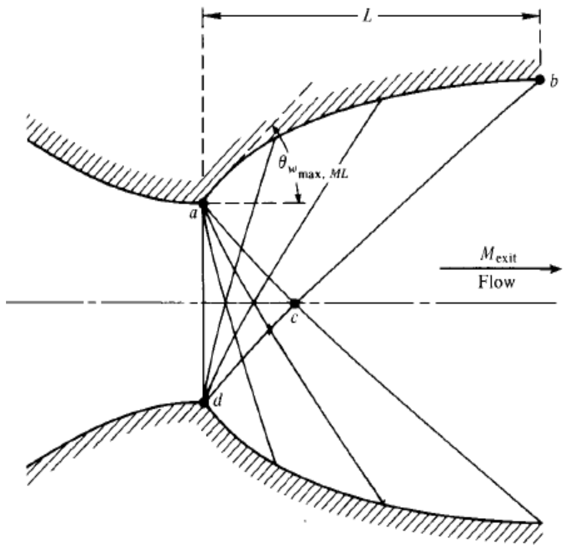

Fig.4 Minimum length Supersonic Nozzle Design

At point c, from Eq. (2.3),

$$\theta_c + \nu_c = (K_-)_c \qquad \text{---- 4.1}$$

However, $\theta_c = 0$ and $\nu_c = \nu_M$. Hence, from the above equation

$$(K_-)_c = \nu_M \qquad \text{---- 4.2}$$

At point a, along the same C_ characteristic ac, from Eq. (2.3),

$$\theta_{W_{max},M_L} + \nu_M. = (K_-)_a \qquad \text{---- 4.3}$$

Since the expansion at point a is a Prandtl-Meyer expansion from initially sonic conditions, we know that $\nu_a = \theta_{W_{max},M_L}$. Hence, Eq. (3.3) becomes

$$\theta_{W_{max},M_L} = \frac{1}{2}(K_-)_a \qquad \text{---- 4.4}$$

However, along the same C_ characteristic, $(K_-)_a = (K_-)_c$; hence Eq. (3.4) becomes

$$\theta_{W_{max},M_L} = \frac{1}{2}(K_-)_c \qquad \text{---- 4.5}$$

Combining equations. (3.2) and (3.5), we have

$$\theta_{W_{max},M_L} = \frac{\nu_M}{2} \qquad \text{---- 4.6}$$

Equation (3.6) demonstrates that, *for* a minimum-length nozzle the expansion angle of the wall downstream of the throat is equal to one-half the Prandtl-Meyer function for the design exit Mach number. [2]

4. Results and Discussion

The design of minimum length supersonic nozzle is capable of producing minimum length nozzle by contracting the expansion section. With the contraction of the expansion section, the total length of the nozzle reduces. In the above design, the length of the supersonic nozzle is minimum, since the expansion section is minimum. In fact, the expansion section is contracted to a point at the end of the throat.

Earlier, it was mentioned that the nozzle is being designed for the optimum exit Mach number. It should be recalled that the streamlines are turned away from the axis and, afterwards, they are turned back toward the axis in the diverging section. The turning away of the streamlines occurs at the expansion section, whereas the turning in of the streamlines occurs at the cancellation or the straightening section. The turning away angle is a function of local Mach number. Thereby, it is crystal clear to state that the last local as well as maximum turning away angle plays the vital role to define the exit Mach number.

The minimum length of the nozzle for an exit Mach No. 2.4 is achieved with the characteristic grid generated by 43 characteristic lines. The expansion waves are generated at the throat corner. It should also be mentioned that the throat of the nozzle is 0.025m according to the design assumptions. The expansion waves are completely cancelled at the length of 0.01249m and height of 0.0362m

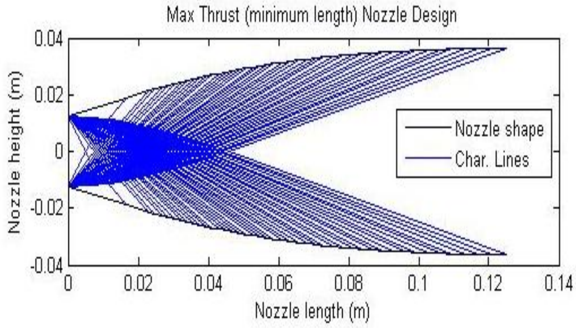

Figure. 4.1. Numerical Output for 43 characteristic Lines

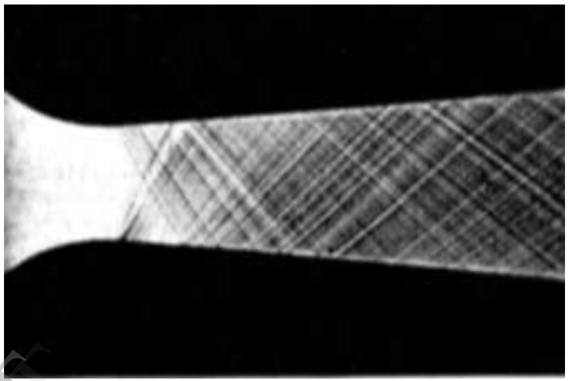

Fig.4.2 Characteristic Lines

With the help of equation (3.0), Mach number distribution can be defined for the exit flow. Mach number distribution is given by the following output:

- For the increase of the characteristic lines from 10 to 200, the nozzle length increases from 0.0856m to 0.0878m and the half of nozzle height increases from 0.0256m to 0.0269m. It has also been observed that the length and height of the nozzle is highly instable at low mesh. Both the length and height become fairly stable as the number of characteristic lines increases.

- The mesh has been made finer by increasing the number of characteristic lines 10 to 200 in five stages. Throughout the mesh increment process, it has become apparent that finer mesh produces rectified result. The evidence of the result rectification is set by nozzle contour smoothening.

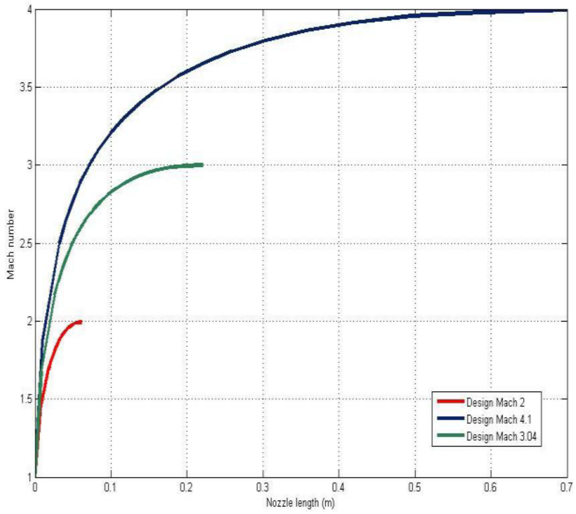

Fig.4.3 Mach number distribution along diverging section.

5. Conclusion

Supersonic nozzles have many applications. They are usually subjected to complex flow pattern. Computer is a must to achieve high accuracy and large calculation needed with modern high speed applications. Hence, a computerized approximation approach might be a better method to tackle such a problem. Characteristic method is the most appropriate method to be used with the supersonic nozzle design. To address the actual conditions, the consideration should be made on account of the phenomena due to the viscous effect, pressure difference with respect to the back pressure, heat conduction and so on. The design presented here can be utilized to compare with the other nozzle designs regarding to the specific design conditions. The simulation program of the practical supersonic nozzle can be developed depending on this design with the considerations of the losses take place in real time.

What this project has done is, it has examined the essential feature required to achieve steady, sustained supersonic flow. That feature is the nozzle contour. A profound appreciation for the process by which the desired exit Mach number is achieved.

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