

Finite Difference and Crank-Nicolson Methods for the Heat Equation

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Abstract

This case study explores the application of numerical methods, specifically the finite difference method and the Crank-Nicolson method, to solve parabolic partial differential equations (PDEs) in various real-world scenarios, including HVAC systems. The focus lies on understanding the principles behind these methods and their practical utility in addressing complex physical phenomena, such as heat distribution and diffusion. The Crank-Nicolson method, in particular, offers a robust approach to solving PDEs by discretizing both time and space dimensions, leading to a stable and accurate numerical solution. By leveraging this method, we aim to model and analyse dynamic systems, including those encountered in HVAC engineering, where precise control of temperature distribution is critical for ensuring occupant comfort and energy efficiency.

1. Introduction

Heating, Ventilation, and Air Conditioning (HVAC) systems play a pivotal role in maintaining indoor comfort and air quality in buildings, ranging from residential homes to large commercial structures. These systems regulate temperature, humidity, and air circulation to create optimal indoor environments conducive to human health, productivity, and well-being. However, achieving efficient and effective HVAC operation requires

sophisticated engineering solutions, often involving complex mathematical modelling and simulation techniques. One such technique widely employed in HVAC engineering is the Crank-Nicolson method, a numerical approach for solving parabolic partial differential equations (PDEs). In the context of HVAC systems, the Crank-Nicolson method finds application in modelling heat transfer processes, which are fundamental to understanding and optimizing thermal comfort within indoor spaces.

The Crank-Nicolson method offers several advantages for simulating heat distribution in HVAC systems. Unlike explicit methods, which may suffer from stability issues and restrictive time step sizes, Crank-Nicolson provides a stable and efficient numerical solution by discretizing both time and space dimensions. This approach allows for the accurate prediction of temperature profiles and heat flow dynamics over time, enabling engineers to design HVAC systems that meet stringent performance requirements while minimizing energy consumption.

2. Problem Statement

The problem at hand concerns maintaining optimal indoor conditions in an office building for HVAC users. The challenge is to regulate temperature effectively while minimizing energy consumption. Utilizing the Crank-Nicolson method, our objective is to predict temperature dynamics accurately over time, ensuring occupant comfort and energy efficiency.

3. Basic concepts related to the topic

The basic concepts related to this case study project include understanding the heat equation as a parabolic PDE that models heat transfer phenomena. The heat equation describes how the temperature distribution in a given region changes over time due to heat conduction. Additionally, the finite difference method is introduced as a numerical technique for approximating derivatives using discrete differences.

Suppose we have a function $(x; y; z; t)$, which describes the temperature of a conducting material at a given location, $(x; y; z)$, you can use this function to determine the temperature at any position on the material at a future time, $t + 1$. The function U changes over time as heat spreads through-out the material and the heat equation is used to determine this change in the function U . The gradient of U describes which direction and at what rate is the

temperature changing around a particular region of the material. This gradient will help us determine the heat through various materials. The heat equation is a parabolic partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time.

The heat equation, a classic example of a parabolic PDE, is mathematically represented as:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Where $u(x,t)$ denotes the temperature distribution as a function of spatial variable x and time t , and α represents the thermal diffusivity of the material.

Parabolic PDEs in general have the form:

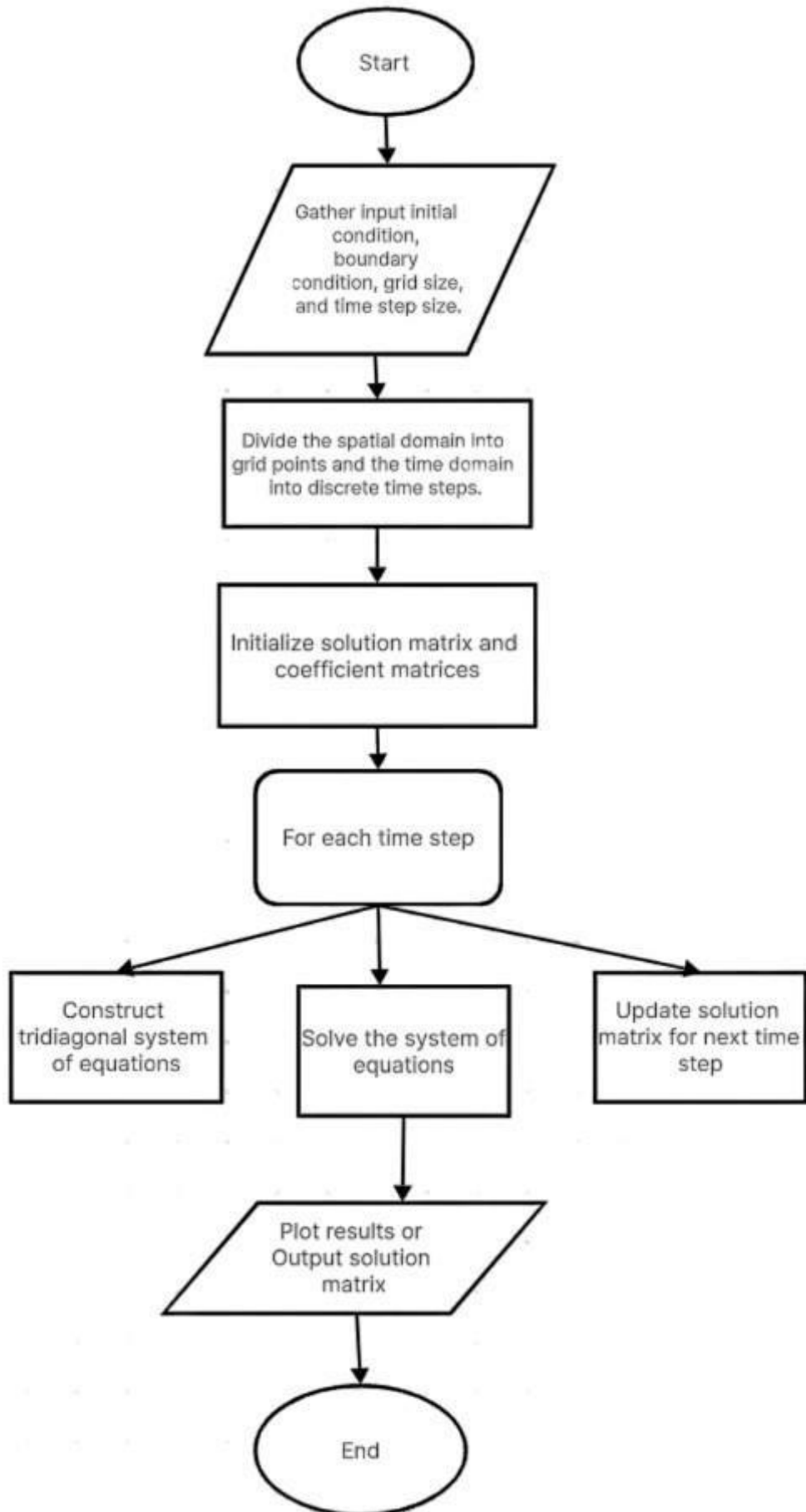
$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) + f$$

D is a diffusion coefficient matrix and f is source.

The finite difference method involves discretizing the spatial and temporal domains of the PDE to approximate derivatives using finite differences. For the heat equation, a typical discretization involves dividing the spatial domain into a grid and approximating the second derivative in the heat equation using central differences. The Crank-Nicolson method is a numerical scheme used to solve parabolic PDEs by combining forward and backward time steps to improve accuracy and stability. It is based on approximating the spatial derivatives using central differences at the midpoints between time steps.

4. Flowchart

The following flowchart illustrates the step-by-step process involved in implementing the Crank-Nicolson method for solving parabolic partial differential equations (PDEs).



5. Software/Hardware used

Operating System: Windows 11

Toolbox: None

Hardware: Personal Computer with Intel Core i7 Processor and 8GB RAM

Software: Scilab Version: 6.1.1

6. Procedure of execution

Step 1: Launch the Scilab software on your computer.

Step 2: Open the provided Scilab script file named "heat__equation1.sce".

Step 3: Execute the script by navigating to the "Execute" menu and selecting "**File with no echo**".

Step 4: Enter input values as required when prompted by the script.

Step 5: Observe the graph and console output to analyse the temperature distribution and numerical simulation results.

7. Result

The user ran the code and gave the following input:

- **Initial position along the x-axis:** This parameter defines the starting point along the horizontal axis (x-axis) where temperature measurements or simulations will begin within the building. In an HVAC scenario, this could represent the location within the building where temperature monitoring or control systems are installed, typically at one end of the building.

Input: 0 (in meters)

- **Final position along the x-axis:** This parameter indicates the endpoint along the horizontal axis (x-axis) where temperature measurements or simulations will end within the building. In HVAC applications, it represents the opposite end of the building from the initial position, completing the spatial domain where temperature monitoring or control is considered.

Input: 1 (in meters)

- **Step size for x:** It defines the distance between adjacent measurement points along the horizontal axis (x-axis) within the building. In HVAC systems, this parameter determines the spatial resolution of temperature monitoring or simulation, allowing for finer or coarser measurements based on the building's size and complexity.

Acceptable range: Positive real numbers.

Boundary: Should not be zero;

Input: 0.1 (in meters)

- **Initial time:** This parameter specifies the starting time for the simulation of temperature dynamics within the building. In HVAC scenarios, it represents the beginning of the simulation period, often aligned with the start of occupancy or system activation.

Input: 0 (in seconds)

- **Time step size:** It indicates the duration of each time step used in the simulation, determining the temporal resolution of the temperature dynamics. For HVAC simulations, this parameter controls how frequently temperature changes are calculated and updated over time, influencing the accuracy and computational efficiency of the simulation.

Acceptable range: Positive real numbers.

Boundary: Should not be zero;

Input: 0.005 (in seconds)

- **Number of time steps:** This parameter defines the total number of time steps or iterations to be simulated during the entire duration of the HVAC system's operation or the monitoring period. In HVAC applications, it determines the overall duration of the simulation and the granularity of temporal information captured, reflecting the system's response to various conditions over time.

Acceptable range: Positive integers.

Boundary: Should be a whole number

Input: 20 (dimensionless)

Figure A

```
Enter initial position along the x-axis (starting point of the length of the building) (in meters): 0
Enter final position along the x-axis (endpoint of the length of the building) (in meters): 1
Enter step size for x (distance between discrete measurement points along the x-axis) (in meters): 0.1
Enter initial time (starting point of the simulation) (in seconds): 0
Enter time step size (duration of each time step) (in seconds): 0.005
Enter number of time steps (total duration of the simulation) (dimensionless): 20
```

Matrix A represents the coefficients used in the finite difference approximation for calculating the second derivative with respect to space, while Matrix B represents the coefficients used in the implicit time-stepping scheme.

Figure B

```
"Matrix a:"

 3.   -0.5  0.   0.   0.   0.   0.   0.   0.
-0.5  3.   -0.5  0.   0.   0.   0.   0.   0.
 0.   -0.5  3.  -0.5  0.   0.   0.   0.   0.
 0.   0.   -0.5  3.  -0.5  0.   0.   0.   0.
 0.   0.   0.   -0.5  3.  -0.5  0.   0.   0.
 0.   0.   0.   0.   -0.5  3.  -0.5  0.   0.
 0.   0.   0.   0.   0.   -0.5  3.  -0.5  0.
 0.   0.   0.   0.   0.   0.   -0.5  3.  -0.5
 0.   0.   0.   0.   0.   0.   0.   -0.5  3.

"Matrix b:"

 1.   0.5  0.   0.   0.   0.   0.   0.   0.
 0.5  1.   0.5  0.   0.   0.   0.   0.   0.
 0.   0.5  1.   0.5  0.   0.   0.   0.   0.
 0.   0.   0.5  1.   0.5  0.   0.   0.   0.
 0.   0.   0.   0.5  1.   0.5  0.   0.   0.
 0.   0.   0.   0.   0.5  1.   0.5  0.   0.
 0.   0.   0.   0.   0.   0.5  1.   0.5  0.
 0.   0.   0.   0.   0.   0.   0.5  1.   0.5
 0.   0.   0.   0.   0.   0.   0.   0.5  1.
```

The header "u:" indicates that the following values represent the matrix u. Each row represents a different position along the spatial domain, while each column represents a different time step. The values within the matrix indicate the temperature at each position and time step. The first column represents the temperature distribution at the initial time step, and subsequent columns represent the temperature distribution at subsequent time steps. The matrix is organized in a tabular format, with row and column labels provided for clarity.

For example, the value at row 2 and column 3 (0.2801961) represents the temperature at the second position in the spatial domain at the third time step. Similarly, the value at row 5 and column 15 (0.4792526) represents the temperature at the fifth position in the spatial domain at the fifteenth time step.

Figure C

```

"u:"
      column 1 to 15
0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.
0.309017  0.2942539  0.2801961  0.2668099  0.2540632  0.2419255  0.2303677  0.219362  0.2088821  0.1989029  0.1894005  0.180352  0.1717358  0.1635312  0.1557186
0.5877853  0.5597042  0.5329647  0.5075026  0.483257  0.4601697  0.4381854  0.4172514  0.3973175  0.3783359  0.3602611  0.3430499  0.3266609  0.3110549  0.2961944
0.809017  0.7703667  0.7335629  0.6985174  0.6651462  0.6333693  0.6031105  0.5742973  0.5468606  0.5207347  0.4958569  0.4721676  0.4496101  0.4281303  0.4076766
0.9510565  0.9056204  0.862355  0.8211565  0.7819263  0.7445702  0.7089989  0.6751269  0.6428732  0.6121603  0.5829147  0.5550664  0.5285484  0.5032973  0.4792526
1.      0.9522256  0.9067337  0.863415  0.8221659  0.7828875  0.7454855  0.7098704  0.6759568  0.6436634  0.6129128  0.5836313  0.5557487  0.5291981  0.503916
0.9510565  0.9056204  0.862355  0.8211565  0.7819263  0.7445702  0.7089989  0.6751269  0.6428732  0.6121603  0.5829147  0.5550664  0.5285484  0.5032973  0.4792526
0.809017  0.7703667  0.7335629  0.6985174  0.6651462  0.6333693  0.6031105  0.5742973  0.5468606  0.5207347  0.4958569  0.4721676  0.4496101  0.4281303  0.4076766
0.5877853  0.5597042  0.5329647  0.5075026  0.483257  0.4601697  0.4381854  0.4172514  0.3973175  0.3783359  0.3602611  0.3430499  0.3266609  0.3110549  0.2961944
0.309017  0.2942539  0.2801961  0.2668099  0.2540632  0.2419255  0.2303677  0.219362  0.2088821  0.1989029  0.1894005  0.180352  0.1717358  0.1635312  0.1557186
0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.      0.
      column 16 to 21
0.      0.      0.      0.      0.      0.
0.1482793  0.1411953  0.1344498  0.1280265  0.1219102  0.116086
0.2820439  0.2685695  0.2557387  0.243521  0.2318869  0.2208087
0.3882002  0.3696541  0.3519941  0.3351779  0.3191649  0.303917
0.4563566  0.4345545  0.4137939  0.3940252  0.3752009  0.3572759
0.4798418  0.4569176  0.4350887  0.4143026  0.3945096  0.3756621
0.4563566  0.4345545  0.4137939  0.3940252  0.3752009  0.3572759
0.3882002  0.3696541  0.3519941  0.3351779  0.3191649  0.303917
0.2820439  0.2685695  0.2557387  0.243521  0.2318869  0.2208087
0.1482793  0.1411953  0.1344498  0.1280265  0.1219102  0.116086
0.      0.      0.      0.      0.      0.

```

In Figure D, we plotted a 2D graph representing the temperature distribution along the x-axis at different time steps. The x-axis represents the spatial position within the building, while the y-axis represents the temperature. Each curve on the graph corresponds to a specific time step, indicated by the legend. This visualization allows us to observe how the temperature profile evolves over time.

Figure D

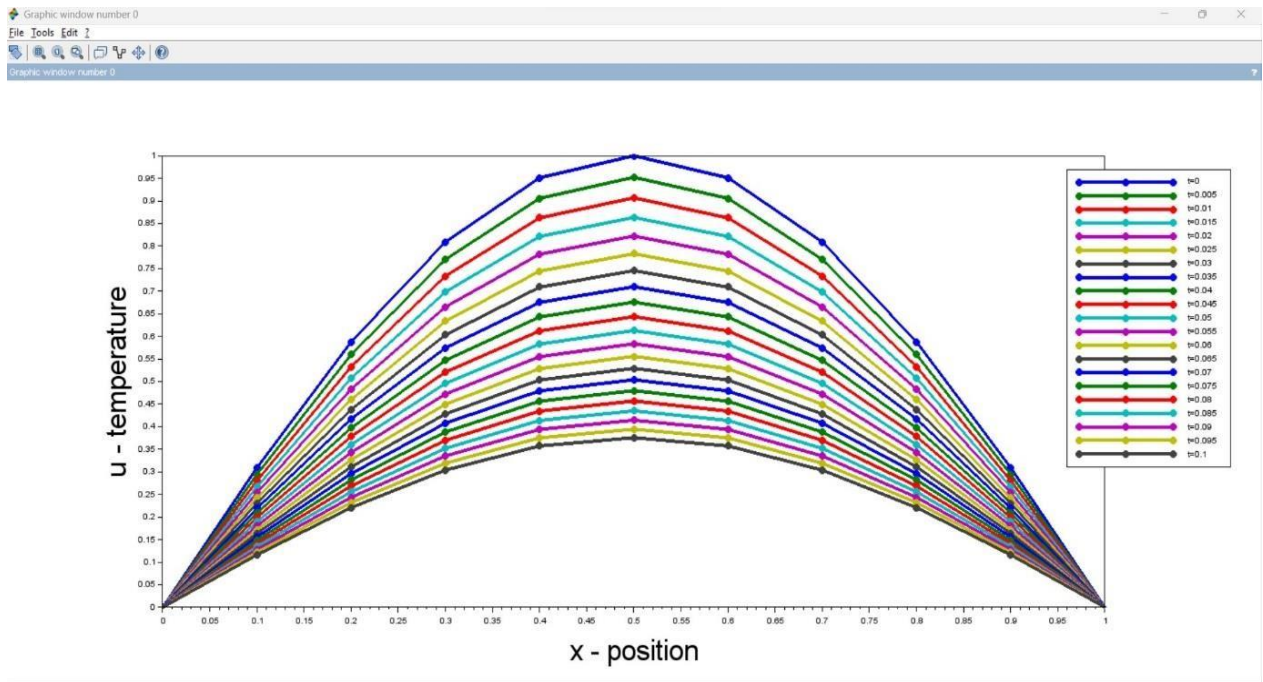
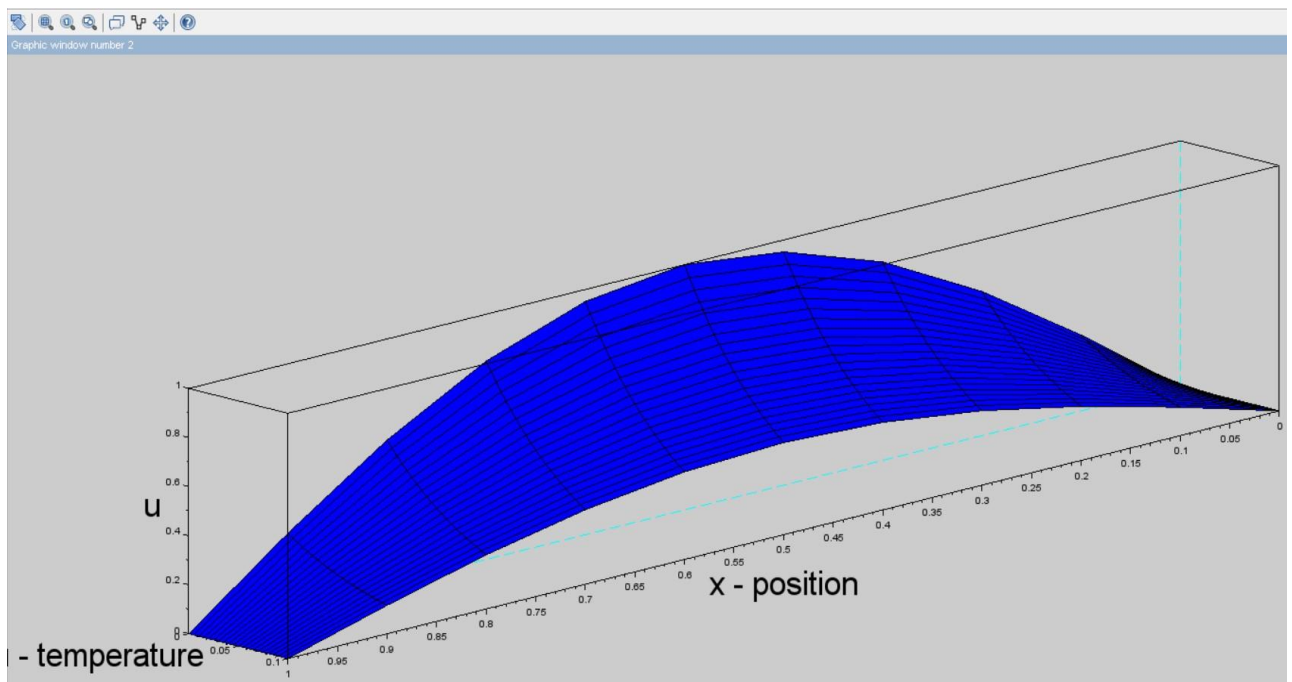


Figure E presents a 3D plot illustrating the temperature distribution within the building over both the spatial and temporal domains.

Figure E



Conclusion: From observing the graphs, it appears that the temperatures is increasing from the edges towards the middle and then decreasing again towards the other edge. This pattern suggests that the temperature is rising towards the centre of the building and then decreasing towards the edges. Such a distribution might be indicative of a heat source located towards the centre of the building or some form of insulation or cooling mechanism near the edges.

Knowing that temperatures are rising towards the centre and decreasing towards the edges allows the HVAC user to adjust the heating and cooling systems accordingly. They may need to increase heating towards the edges or implement zoning strategies to maintain uniform comfort levels throughout the building. By understanding the temperature distribution, the HVAC user can identify areas where heating or cooling is excessive or insufficient. This information enables them to fine-tune the HVAC system to operate more efficiently, reducing energy consumption and associated costs.

8. References

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