National Program on Differential Equations: Theory, Computation & Applications

(NPDE-TCA)
UG Level Training Program
IMA, Bhubaneshwar

(Supported by DST)

Numerical Analysis
(SCILAB)

Lab Sheets

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Lab Sheet-1

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1. Use the **linspace** function to create vectors identical to the following created with colon notation:

   (a) $x = -3 : 100$

   (b) $y = 1 : 0.5 : 20$

2. Let $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$.

   Compute the following using Scilab

   (a) $A + 7\,B$. Is it true $A + 7\,B = 7\,B + A$?

   (b) $A\,B$ and $B\,A$. Is it true $A\,B = B\,A$?

   (c) $B^T\,A^T$. Is it true $(A\,B)^T = B^T\,A^T$?

   (d) $A^T + B^T$. Is it true $(A + B)^T = A^T + B^T$?

3. Consider the following matrices

$$A = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}.$$

   Perform the following operations using Scilab:
   (i) AB (ii) BA (iii) A^T, B^T (iv) $A - B$ (v) A^{-1} and B^{-1}.
   (vi) $\det(A)$ and $\det(B)$ (vii) $\operatorname{rank}(A)$ and $\operatorname{rank}(B)$
   (viii) verify that $(AB)^T = B^T A^T$.

4. Use Scilab to determine all the eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix}.$$

5. Determine the matrix inverse and condition number for the matrix

$$A = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}.$$

6. The following system of equations is designed to determine concentrations (the c's in g/m^3) in a series of coupled reactors as a function of the amount of mass input to each reactor (the right-hand sides in g/day):

$$\begin{aligned} 15c_1 - 3c_2 - c_3 &= 3800, \\ -3c_1 + 18c_2 - 6c_3 &= 1200, \\ -4c_1 - c_2 + 12c_3 &= 2350. \end{aligned}$$

   (a) Express the system of equations in matrix form $A\,X = b$ and determine the matrix inverse A^{-1}.

   (b) Use the inverse to determine the solution.

7. Express the given matrix in LU-factorization form:

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 6 & -4 \\ -1 & -2 & 5 \end{bmatrix}.$$

8. Classify each of the following matrices as strictly diagonally dominant, symmetric positive definite, both, or neither.

$$\text{(i)} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 2 & 0 \\ 4 & 6 & -1 \\ -3 & 2 & 0 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 7 \end{bmatrix}.$$

9. The density of fresh water can be computed as a function of temperature with the following equation

$$\rho = 5.5289 \times 10^{-8}\, T_c^3 - 8.5016 \times 10^{-6}\, T_c^2 + 6.5622 \times 10^{-5}\, T_c + 0.99987,$$

where ρ = density (g/cm^3) and T_c = temperature (0_C).

Use Scilab to generate a vector of temperatures ranging from 32^0F to 82.4^0 F using increments of 3.6^0F. Convert this vector to degree celsius and then compute a vector of densities based on the formula. Generate a plot of ρ versus T_c.

(Hint. $T_c = \frac{5}{9(T_F - 32)}$.)

10. Consider the wind tunnel data for force (F) versus velocity (v):

v (m/s)	10	20	30	40	50	60	70	80
F (N)	15	65	140	440	520	980	790	1360

Use Scilab to create a plot displaying both the data v and F.

11. The following parametric equations generate a helix that contracts exponentially as it evolves

$$x = e^{-2.5t}\sin t, \quad y = e^{-2.5t}\cos t, \quad z = t.$$

Use subplot to generate a two-dimensional line plot of (x, y) in the top and a three-dimensional line plot of (x, y, z) in the bottom.

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Lab Sheet-2

All the problems use Scilab to compute the approximation root.

1. Find an approximation to $\sqrt{3}$ correct to within 10^{-6} using the bisection method.

2. Use bisection method to find the solution accurate to within 10^{-5} for
$$f(x) = 2\sin x - \frac{e^x}{4} - 1,$$
on the interval $[-5, -3]$.

3. Use bisection method to determine the minimum number of iterations necessary to solve
$$f(x) = x^3 + x - 4 = 0$$
with an accuracy 10^{-4} on the interval $[1, 4]$.

4. The equation
$$f(x) = 4\sin x - e^x = 0,$$
has a root on the interval $[0, 0.5]$. Find the root correct to within 10^{-6} using secant method.

5. Let
$$f(x) = x^2 - 6$$
with $p_0 = 2$ and $p_1 = 3$ then find p_7 using secant method.

6. Let
$$f(x) = -x^3 - \cos x,$$
and $p_0 = -1$. Use Newton's method to find p_{10}. Could $p_0 = 0$ be used?

7. Use Newton's method to find the root of the equation

$$f(x) = 4x^3 - 1 - e^{\frac{x^2}{2}} = 0,$$

near $p_0 = 1.0$. Perform 20 iterations.

8. Use Newton's method to find the root of the equation

$$f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21}{125}x\right) = 0,$$

near $p_0 = 2.0$. Perform ten iterations.

9. Do five iterations of Newton's method to solve the system of nonlinear equations

$$\begin{aligned} x^2 + y^2 &= 4, \\ e^x + y &= 1. \end{aligned}$$

Use $\mathbf{X}^{(0)} = [1, \quad -1]^T$.

10. Perform ten iterations of Newton's method for the system of nonlinear equations

$$\begin{aligned} 4\,x_1^2 - x_2^2 &= 0, \\ 4\,x_1\,x_2^2 - x_1 &= 1. \end{aligned}$$

Use $\mathbf{X}^{(0)} = [0, \quad 1]^T$.

11. Do ten iterations of Newton's method to solve the system of nonlinear equations

$$\begin{aligned} x\,y^2 + x^2\,y + x^4 &= 3, \\ x^3\,y^5 - 2\,x^5\,y - x^2 &= -2. \end{aligned}$$

Use $\mathbf{X}^{(0)} = [1, \quad 1]^T$.

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Lab Sheet-3

All the problems use Scilab to perform the computations.

1. Solve the system of linear equations

$$\begin{aligned} 0.7\,x_1 + 1725\,x_2 &= 1739, \\ 0.4352\,x_1 - 5.433\,x_2 &= 3.271, \end{aligned}$$

   using

   (a) Gaussian elimination with no pivoting

   (b) Gaussian elimination with partial pivoting

   Compare the results obtained from each technique with the exact solution $x_1 = 20,\ x_2 = 1$ of the system.

2. Solve the system of linear equations

$$\begin{aligned} 3.41\,x_1 + 1.23\,x_2 - 1.09\,x_3 &= 4.72, \\ 2.71\,x_1 + 2.14\,x_2 + 1.29\,x_3 &= 3.10, \\ 1.89\,x_1 - 1.91\,x_2 - 1.89\,x_3 &= 2.91, \end{aligned}$$

   using

   (a) Gaussian elimination with no pivoting

   (b) Gaussian elimination with partial pivoting

   Compare the results obtained from each technique with the exact solution of the system.

3. Solve the system of linear equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= -2, \\ -x_1 + 2x_3 &= -1, \\ 3x_1 + 2x_2 - x_3 &= 0, \end{aligned}$$

by using Crout decomposition and Doolittle decomposition method.

4. Find the LU-factorization of the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix},$$

in which L is lower triangular and U is an unit upper triangular matrix.

5. Prove that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

does not have an LU-factorization.

6. Are these matrices positive definite?

$$\text{(i)} \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \text{(ii)} \quad \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

7. For what value(s) of α is this matrix positive definite?

$$A = \begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix}.$$

8. Find $\|X\|_\infty$ and $\|X\|_2$ for the following vectors

$$\text{(i)} \quad \left[3,\ -4,\ 0,\ \frac{3}{2}\right]^T, \quad \text{(ii)}\ [2,\ 1,\ -3,\ 4]^T.$$

9. Find $\|\cdot\|_\infty$ for the following matrices

$$\text{(i)} \quad \begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix}, \quad \text{(ii)} \quad \begin{bmatrix} 10 & 0 \\ 15 & 1 \end{bmatrix}.$$

10. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then find $k_\infty(A)$ (condition number of A in maximum-norm).

11. Solve the system of linear equations

$$\begin{aligned} 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4, \\ 3x_1 - x_2 + x_3 &= 1, \end{aligned}$$

using Jacobi iteration method with starting $X^{(0)} = [1,\ 1,\ 1]^T$.

12. Use Gauss-Seidel method to solve the following system of equations

$$\begin{aligned} 4x_1 - 10x_2 + 5x_3 &= 32, \\ 5x_1 - 4x_2 + 10x_3 &= 39, \\ 10x_1 + 5x_2 - 4x_3 &= 17 \end{aligned}$$

with starting initial vector $X^{(0)} = [\ 1,\ \ -1,\ \ 1\ ]^T$.

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Lab Sheet-4

All the definite integrals, find the approximate value using Scilab.

1. Use the composite Trapezoidal rule to approximate the following integrals
$$\text{(i)} \quad \int_1^2 x\,\ln(x)\,dx, \quad \text{(ii)} \quad \int_{-2}^2 x^3\,e^x\,dx$$
with $n = 10$ equal parts.

2. Approximate the following integral
$$\int_0^2 x^2\,e^{-x^2}\,dx,$$
using the composite Trapezoidal rule and the composite Simpson's rule with $h = 0.01$.

3. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 4 and Simpson's rule gives the value 2. What is $f(1)$?

4. Determine the number of subintervals N so that the composite Trapezoidal rule give the value of the integral
$$\int_0^1 \frac{1}{1+x^2}\,dx,$$
correct up to 4-decimal digits.

5. Use Simpson's $\frac{1}{3}$rd rule to approximate the integral
$$\int_0^1 \frac{1}{1+x}\,dx,$$
with 4 equal subintervals.

6. Given the function f at the following values

x	1.8	2.0	2.2	2.4
$f(x)$	3.1213	4.4214	6.0424	8.0302

approximate $\int_{1.8}^{2.4} f(x)\,dx$ using $\frac{3}{8}$th Simpson's rule.

7. Evaluate the integral
$$\int_{-1}^{1} \frac{x\,\sin x}{(1+x^2)}\,dx$$
by using 2-point Gauss-Legendre quadrature rule.

8. Evaluate the integral
$$\int_{0}^{1} e^{-x^2}\,dx,$$
by using 2-point Gauss-Legendre quadrature rule.

9. Evaluate the integral
$$\int_{1}^{4} \frac{xe^{2x}}{1+x^2}\,dx,$$
by using 3-point Gauss-Legendre quadrature formula.

10. Use three-point Gauss-Legendre quadrature formula to evaluate the integral
$$\int_{2}^{3} \frac{\cos 2x}{1+\sin x}\,dx.$$

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Lab Sheet-5

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1. The height of the fluid in a tank ($h(t)$) whose outlet flow is dependent on the pressure head (height of fluid) inside the tank and whose inlet flow is a function of time may be modeled via the equation

$$\frac{dh}{dt} = \alpha(t) - \beta\sqrt{h}, \quad h(0) = h_0.$$

   Use Scilab ODE function to find the solution $h(t)$ for $0 < t < 30$ if the following values for the parameters are given. Input flow: $\alpha(t) = 10 + 4\sin(t)$, $\beta = 2$, $h_0 = 1$.

   Also find the approximation solution using

   (a) Forward Euler

   (b) Modified Euler

   (c) 2nd order Runge-Kutta method

   (d) 4th order Runge-Kutta method

2. Consider the initial value problem (IVP)

$$\begin{aligned} y'(t) &= t\sin\left(\pi y(t)\right), \\ y(1) &= 0.5. \end{aligned}$$

   Use Scilab ODE function to find the solution at $y(1.5)$ with $n = 2, 4, 8, 16$ partitions. Plot the solution against t. Also find the approximation solution using

   (a) Forward Euler

   (b) Modified Euler

(c) 2nd order Runge-Kutta method

(d) 4th order Runge-Kutta method

3. Consider the Predator-Prey equation

$$\begin{aligned} u'(t) &= (2 - v(t))\, u(t), \\ v'(t) &= (u(t) - 1)\, v(t) \end{aligned}$$

with the initial conditions $u(0) = 3, v(0) = 2$.

(a) Use Scilab ODE function to solve the problem with in $0 \le t \le 10$, using $n = 100$ steps.

(b) Give two graphs showing the curves $u(t)$ and $v(t)$ against t and a plot of $u(t)$ against $v(t)$.

4. A mass-spring system can be modeled via the following second-order ODE

$$y'' + cy' + \omega^2 y = g(t), \quad y(0) = 1,\ y'(0) = 0.$$

Find the numerical solution in the time interval $0 \le t \le 10$ for the particular set of conditions $c = 5,\ \omega = 2$ and $g(t) = \sin(t)$ using Scilab.

5. Consider the simple motion of pendulum. Let $\theta(t)$ be the angle made with the vertical at time t. Initially $\theta(0) = \pi/4$ and $\theta'(0) = 0$. The subsequent position is described by the differential equation

$$\theta''(t) - \frac{g}{l} \sin(\theta(t)) = 0,$$

where $g = 9.8m/sec^2$ is the acceleration due to gravity and $l = 0.5$m is the length of the pendulum.

(a) Solve for $0 \le t \le 5$ by taking 40 steps using Scilab ODE function. Plot $\theta(t)$. From this computation, what can you conclude about the pendulum's amplitude as time increases.

(b) Repeat using 400 steps. What is your estimate of the period of the pendulum's movement.

6. (Vander Pol's equation) The following equation describes the voltage across the triode circuit:

$$\begin{aligned} v''(t) + \epsilon \left(v(t)^2 - 1\right) v'(t) + v(t) &= 0, \\ v(2) &= 1, v'(2) = 0. \end{aligned}$$

Take $\epsilon = 0.897$ and compute the approximate solution at $t = 10$ using Scilab ODE function. Plot the numerical solution.

7. Consider a nonlinear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical systems

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y^3 = \gamma\cos(\omega t).$$

   (a) Set the parameter values to $\epsilon = 0.15$, $\gamma = 0.3$ and $\omega = 1$.

   (b) Use the initial conditions: $y(0) = -1$, $y'(0) = 1$.

   (c) Use Scilab ODE function to solve on the t interval $[0, 100]$.

   (d) Plot the solution $y(t)$.

8. Consider the Lorenz system of equations

$$\begin{aligned}
\frac{dx_1}{dt} &= \sigma\left(x_2 - x_1\right), \\
\frac{dx_2}{dt} &= \left((1+r) - x_3\right) x_1 - x_2, \\
\frac{dx_3}{dt} &= x_1x_2 - bx_3
\end{aligned}$$

   with the initial conditions

$$x_1(0) = -10, \;\; x_2(0) = 10, \;\; x_3(0) = 25.$$

   (a) Use $\sigma = 10$, $r = 28$, $b = 8/3$.

   (b) Use Scilab ODE function to solve on the t interval $[0, 50]$.

   (c) Plot the Lorenz attractor.

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Lab Sheet-6

All the boundary value problems use central finite difference scheme and use Scilab to compute the approximate solutions.

1. Consider the following boundary value problem (BVP)

$$\begin{aligned} -u'' + \pi^2 u &= 2\pi^2 \sin(\pi x),\ 0 < x < 1, \\ u(0) &= 0, u(1) = 0. \end{aligned}$$

   The exact solution is given by

$$u(x) = \sin(\pi x).$$

   Plot the numerically computed solution and the exact solution. Do the grid refinement analysis to determine the order of accuracy of the global solution.

2. Consider the following boundary value problem (BVP)

$$\begin{aligned} u'' + (x+1)u' - 2u &= (1-x^2)\, e^{-x},\ 0 < x < 1, \\ u(0) &= -1,\ u(1) = 0. \end{aligned}$$

   The exact solution is given by

$$u(x) = (x-1)\, e^{-x}.$$

   Plot the numerically computed solution and the exact solution. Do the grid refinement analysis to determine the order of accuracy of the global solution.

3. Consider the finite difference scheme for one dimensional steady state convection-diffusion equation

$$\epsilon u'' - u' = -1,\ \ 0 < x < 1,$$
$$u(0) = 1,\ \ u(1) = 3.$$

(i) Verify the exact solution is

$$u(x) = 1 + x + \left(\frac{e^{\frac{x}{\epsilon}} - 1}{e^{\frac{1}{\epsilon}} - 1} \right).$$

(ii) Compare the following two finite difference methods for $\epsilon = 0.3, 0.1, 0.05$ and 0.0005,

(a) central difference scheme:

$$\frac{\epsilon\,(U_{i-1} - 2U_i + U_{i+1})}{h^2} - \frac{U_{i+1} - U_{i-1}}{2h} = -1,$$

(b) central-upwind difference scheme:

$$\frac{\epsilon\,(U_{i-1} - 2U_i + U_{i+1})}{h^2} - \frac{U_i - U_{i-1}}{h} = -1.$$

Plot the numerically computed solution and the exact solution for $h = 0.1$, $h = \frac{1}{25}$ and $h = 0.01$.

Do grid refinement analysis for each case to determine the order of accuracy.

Use Scilab command subplot to put several graphs together.

(c) From your observation, give your opinion to see which method is better.

4. Write a Scilab program to find the numerical solution of the following boundary value problem (BVP)

$$u'' + u = \sin 3x, \quad x \in \left[0, \frac{\pi}{2}\right],$$
$$u(0) + u'(0) = -1, \quad u'(\frac{\pi}{2}) = 1.$$

The exact solution is given by

$$u(x) = -\cos x + \frac{3}{8}\sin x - \frac{1}{8}\sin 3x.$$

Plot the numerically computed solution and the exact solution. Do the grid refinement analysis to determine the order of accuracy of the global solution.

5. Consider the following boundary value problem (BVP)

$$u'' + u = \sin 3x, \quad x \in \left[0, \frac{\pi}{2}\right],$$
$$u'(0) = 1, \; u(\pi/2) + u'(\pi/2) = -1.$$

The exact solution is given by

$$u(x) = \frac{11}{8}\sin x + \frac{5}{2}\cos x - \frac{1}{8}\sin 3x.$$

Plot the numerically computed solution and the exact solution. Do the grid refinement analysis to determine the order of accuracy of the global solution.

6. Consider the following boundary value problem (BVP)

$$u'' - u = 1,\ 0 < x < 1,$$
$$u(0) = 0,\ u(1) + u'(1) = 1.$$

The exact solution is given by

$$u(x) = e^{x-1} + \left(1 - \frac{1}{e}\right)e^{-x} - 1.$$

Plot the numerically computed solution and the exact solution. Do the grid refinement analysis to determine the order of accuracy of the global solution.

Numerical Analysis (AAOC C341) Formula Sheet

1. Secant Method:

$$p_{n+1} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}, \quad n = 1, 2, 3, \cdots,$$

2. Newton's Method:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad n = 0, 1, 2, \cdots,$$

3. Newton's Method for Multiple Roots:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots, \quad \text{for } f(x) = (x-r)^m\, h(x),\ h(r) \neq 0.$$

4. Fixed Point Iteration:

$$p_{n+1} = g\left(p_n\right), \quad n = 0, 1, 2, \cdots.$$

5. Newton's Method for system of equations: $f_1(\mathbf{x}) = 0,\ f_2(\mathbf{x}) = 0, \cdots, f_n(\mathbf{x}) = 0$:

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta\mathbf{x},\ n = 0, 1, 2, \cdots, \text{ where } \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \Delta\mathbf{x} = -\begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}.$$

6. Norms for $\mathbf{x} \in \Re^n$:

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{n} |\mathbf{x}_j|, \qquad \|\mathbf{x}\|_\infty = \max_{j=1}^{n} |\mathbf{x}_j|, \qquad \|\mathbf{x}\|_2 = \left(\sum_{j=1}^{n} \mathbf{x}_j^2\right)^{1/2}.$$

7. Norms for $A \in \Re^{n\times m}$:

$$\|A\|_1 = \max_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}|, \qquad \|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|.$$

8. The Frobenius norm for $A \in \Re^{n\times m}$ is defined as

$$\|A\|_f = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2\right)^{1/2}.$$

9. Condition Number:

$$k(A) = \|A\| \, \|A^{-1}\|.$$

10. Lagrange Interpolation for points $(x_i, f_i),\ i = 0, 1, 2, \cdots, n$;

$$l_i(x) = \Pi_{j=0,\, j\neq i}^{n} \ \frac{x - x_j}{x_i - x_j}, \quad p_n(x) = \sum_{i=0}^{n} l_i(x)\, f_i.$$

11. Divided differences: example of $f\,[x_0, x_1, x_2] = \frac{f[x_0,x_1] - f[x_1,x_2]}{x_0 - x_2}$:

$$\begin{aligned} p_n(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots \\ &+ f[x_0, x_1, x_2, \cdots, x_n]\,(x - x_0)\,(x - x_1)\,(x - x_2) \cdots (x - x_{n-1}). \end{aligned}$$

12. Interpolation Error for $p_n(x)$:

$$E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\,(x - x_1)\ \cdots (x - x_n).$$

13.

$$\begin{aligned} u'(x) &\approx \frac{\Delta u(x)}{h} = \frac{u(x+h) - u(x)}{h}, \quad \text{(forward difference formula)} \\ u'(x) &\approx \frac{\nabla u(x)}{h} = \frac{u(x) - u(x-h)}{h}, \quad \text{(backward difference formula)} \\ u'(x) &\approx \frac{\delta u(x)}{2h} = \frac{u(x+h) - u(x-h)}{2h}, \quad \text{(central difference formula)} \end{aligned}$$

14. Central difference formula:

$$u''(x) \ = \ \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2), \quad h > 0.$$

15. Trapezoidal rule:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\,[f(x_0) + f(x_1)] - \frac{h^3}{12}\,f''(\xi), \ \text{ where } \ x_0 < \xi < x_1.$$

16. Simpson's $\frac{1}{3}$rd rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\,[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}\,f^{(4)}(\xi), \quad \text{where } \ x_0 < \xi < x_2.$$

17. Simpson's $\frac{3}{8}$th rule:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\,[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80}\,f^{(4)}(\xi), \quad \text{where } \ x_0 < \xi < x_3.$$

18. Gaussian Quadrature (n-point) rule:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=1}^{n} w_j\, f(\xi_j),$$

where w_j are weights and ξ_j are Gaussian points.

19. Single step methods to solve the initial value problem (IVP):

$$\begin{aligned} \frac{dy}{dx} &= f(x, y(x)), \\ y(x_0) &= y_0. \end{aligned}$$

(a) Forward Euler's method:

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, 2, \cdots$$

(b) Backward Euler's method:

$$y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1}), \quad n = 0, 1, 2, \cdots$$

(c) Modified Euler's method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right], \quad n = 0, 1, 2, \cdots,$$

here $y^*_{n+1} = y_n + h\, f(x_n, y_n)$.

(d) A second-order Runge-Kutta Method:

$$y_{n+1} = y_n + \frac{h}{2}\left(k_1 + k_2\right), \quad n = 0, 1, 2, \cdots$$

where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + h, y_n + h\, k_1)$.

(e) A fourth-order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2\, k_2 + 2\, k_3 + k_4\right), \quad n = 0, 1, 2, \cdots$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), & k_2 = f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \\ k_3 &= f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}), & k_4 = f(x_n + h, y_n + k_3). \end{aligned}$$

20. Multi step methods to solve the initial value problem (IVP):

$$\begin{aligned} \frac{dy}{dx} &= f(x, y(x)), \\ y(x_0) &= y_0. \end{aligned}$$

(a) Two step Adams-Bashforth method:

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right], \quad n = 1, 2, 3, \cdots$$

(b) Four step Adams-Bashforth method:

$$y_{n+1} = y_n + \frac{h}{24}\left[55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})\right],$$

where $n = 3, 4, 5, \cdots$

(c) Three step Adams-Moulton method:

$$y_{n+1} = y_n + \frac{h}{24}\left[9f(x_{n+1}, y_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right],$$

where $n = 2, 3, 4, \cdots$

(d) Adams-Moulton Predictor-Corrector formula:

$$\begin{aligned} y^*_{n+1} &= y_n + \frac{h}{24}\left[55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})\right], \\ y_{n+1} &= y_n + \frac{h}{24}\left[9f(x_{n+1}, y^*_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right]. \end{aligned}$$

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