Fractional Calculus and Fractional Differential Equations with SCILAB

Scilab and Its Applications to Global Optimization and Fractional Differential Equations

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Outline

- What is Fractional Calculus
- Historical Review
- Definitions
- Applications
  - Fractional-order modeling
  - Fractional-order control

What is Fractional Calculus

- All of us are familiar with *normal* derivatives and integrals, like, $\frac{df}{dt}$, $\frac{d^2f}{dt^2}$, $\int_0^t f(u)du$.
- We have first-order, second-order derivatives, or first integral, double integral, of a function.
- Now we wish to have half-order, πth-order, or ...... (3-6i)th-order derivative of a function.
- So, Fractional calculus $\Rightarrow$ derivatives and integrals of arbitrary real, or complex order

So what? Answer these questions.....

- Does it make any sense? Or is just a mathematical fantasy? Define it.
- Tell me how to calculate the $1/2$-order derivative of $f(t) = t$.
- *This seems to be a recent stuff*. How old is it?
- Does it have any physical interpretation/geometrical meaning?
- Why study it? How is it important in engineering? What's the deal?
- How much serious is the research community about it?
- What are its applications?
- So should we discard the integer-order derivatives?

History of Fractional calculus

- As old as normal, conventional, integer-order calculus.
- Born in **1695**!!
- In a letter correspondence, l'Hôpital asked Leibniz: "What if the order of the derivative is $1/2$"?
- To which Leibniz replied in a prophetical way, "Thus it follows that will be equal to $x^2 \sqrt[2]{dx : x}$, an apparent paradox, from which one day useful consequences will be drawn."
- This letter of Leibniz was dated 30th September, 1695. So **30th September** is considered as the **birthday** of fractional calculus.

Contributors

- Leibniz (1695)
- Euler (1730)
- Lagrange (1772)
- Laplace (1812)
- Fourier (1822)
- Abel (1823)
- Liouville (1832)
- Riemann (1876)

Acceptance by Research Community

- **Fifteen** Books.
- **Two** dedicated international journals.
- **First** international conference on "Fractional Calculus and its Applications" in June, 1974 in US.
- Special international conference conducted (first was in 2004) by the International Federation of Automatic Control (IFAC) every two years: **Fractional Differentiation and its Applications**.
- More than 5000 papers published on the single topic of modeling of complex systems by fractional differential equations.

Indians Working in Fractional Calculus

1. Prof. H. M. Srivastava, University of Victoria, Canada
2. Prof. Loknath Debnath, The University of Texas-Pan American, US
3. Prof. Lakshmikantham, Florida Institute of Technology, US
4. Prof. Gangal, University of Pune
5. Prof. Saxena, Jai Narain Vyas University, Jodhpur
6. Prof. Mathai, McGill University, Canada
7. Prof. Arun Kolwankar, Ramniranjan Jhunjhunwala college, Mumbai
8. Mr. Shantanu Das, Scientist, Bhabha Atomic Research Center, Mumbai
9. Prof. Anindya Chatterjee, Indian Institute of Science, Bangalore
10. Prof. S. Sen, IIT Kharagpur
11. Prof. Arijit Biswas, Jadavpur University, Kolkata
12. Prof. (Mrs.) Varsha Gejji, University of Pune

Some facts about Fractional Calculus

- Rigorous mathematical theory has been developed.
- Integer-order calculus is the special case.
- Geometrical interpretation or physical meaning exists. But not as straight forward as for the integer-order derivatives.
- There are more that FIFTEEN definitions of fractional derivative operator.

Some Mathematics: Successive Differentiation

- nth-order derivative of t^n (n is integer)

$$\frac{d^n}{dt^n} t^n = n!$$

- nth-order derivative of t^m, m, n integers, $m > n$ is

$$\frac{d^n}{dt^n} t^m = \frac{m!}{(m-n)!} t^{m-n}$$

- Use the Euler's Gamma function (Γ) property,

$$n! = \Gamma(n+1)$$

- So we can re-write

$$\frac{d^n}{dt^n} t^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}$$

- Gamma function is defined for positive and negative reals (except for negative integers and zero). So we let m and n to be reals.

Some more Mathematics

- We define fractional derivative of order $\alpha \in \mathbf{R}$, $\alpha \geq 0$, of t^{μ}, $\mu \in \mathbf{R}$:

$$\frac{d^{\alpha}}{dt^{\alpha}} t^{\mu} := D_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}$$

- Here the condition $\mu > \alpha$ can be relaxed.
- Let's Start!!

1. $$\frac{d^{0.5}}{dt^{0.5}} t^{0.5} = \Gamma(1+0.5) = \Gamma(1.5)$$

2. $$\frac{d^{0.5}}{dt^{0.5}} t = \frac{1}{\Gamma(1.5)} t^{0.5}$$

3. $$\frac{d^{0.5}}{dt^{0.5}}(1) = \frac{1}{\sqrt{\pi t}} \neq 0$$

Some more Mathematics: Successive Integration

- Let's take the first integral a function $f(t)$

$$J^1 f(t) := \int_0^t f(u)du, \quad t > 0,$$

- Let's integrate it once more.

$$J^2 f(t) := \int_0^t \int_0^u f(v)dvdu, \quad t > 0,$$

- Successive integration of $f(t)$ for n-times (n, integer) is:

$$J^n f(t) := \int_0^t \int_0^u \dots \int_0^w f(v)dvdw \dots du, \quad t > 0,$$

- Cauchy provided a closed-form formula for n successive integrations:

$$J^n f(t) := \frac{1}{(n-1)!} \int_0^t (t-u)^{(n-1)} f(u)du, \quad t > 0.$$

- Again, the **same** trick!! Replace factorial by Gamma function.

Some more.....

- The **Riemann-Liouville** (RL) fractional integral of order $0 \leq \alpha \leq 1$ is defined as

$$J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{(\alpha-1)} f(u) du, \quad t > 0,$$

- We can define fractional derivative of order α by two ways:
  1. **RL** fractional derivative: Take fractional integral of order $(1-\alpha)$ and then take a first derivative,

$$D_t^{\alpha} f(t) = \frac{d}{dt} J^{1-\alpha} f(t)$$

  2. **Caputo** fractional derivative: Take first order derivative and then take a fractional integral of order $(1-\alpha)$,

$$D_t^{\alpha} f(t) = J^{1-\alpha} \frac{d}{dt} f(t)$$

Did you notice?

- Definition of fractional derivative involves an integration.
- Integration is a non-local operator (as it is defined on an interval).
- $\Rightarrow$ **Fractional derivative is a non-local operator**.
- $\Rightarrow$ Calculating time-fractional derivative of a function $f(t)$ at some $t = t_1$ requires all the past history, i.e. all $f(t)$ from $t = 0$ to $t = t_1$.
- $\Rightarrow$ Fractional derivatives can be used for modeling systems with memory.
- $\Rightarrow$ Calculating space-fractional derivative of a function $f(x)$ at $x = x_1$ requires all non-local $f(x)$ values.
- $\Rightarrow$ Fractional derivatives can be used for modeling distributed parameter systems.

Fractional Differential Equations (FDEs)

- Differential equations involving fractional derivatives.
- Example: Bagley-Torvik equation of oscillatory processes with fractional damping:

$$\frac{d^2}{dt^2}y(t) + aD_t^{1.5}y(t) + by(t) = f(t)$$

- Both ODEs and PDEs.
- Linear and non-linear.
- Existence and uniqueness of solutions established.
- Analytical solutions are difficult to evaluate.
- Dedicated, elegant numerical methods exist.

Applications

- In spite of its long history, fractional calculus was not considered eligible for any applications.
- This was due to its high complexity and lack of physical and geometric interpretation.
- Application of fractional calculus to real-world problems is only four decades old.
- Applications can be broadly categorized into:
  1. **Modeling of Systems**
  2. **Fractional-order Control**

Diffusion

- Normal, **Fickian** diffusion $\Rightarrow$ flow of particles from high concentration to low concentration $\Rightarrow$ Concentration is given by **Gaussian distribution**
- Asymptotical mean-squared displacement is a **linear function** of time,

$$\langle x^2(t) \rangle \sim t$$

- Model is given by diffusion equation

$$\frac{\partial \phi(x,t)}{\partial t} = D \frac{\partial^2 \phi(x,t)}{\partial x^2}$$

Diffusion (continued)

- Some processes are an exception to this.
- Example: Photocopy machine and Laser printer. Movement of holes and electrons in the semiconductors inside them is not the normal, Gaussian diffusion.
- It is the **Anomalous diffusion**.
- Asymptotical mean-squared displacement is **not** a linear function of time,

$$\langle x^2(t)\rangle \sim t^\alpha, \quad \alpha \neq 1$$

- $\alpha < 1 \Rightarrow$ Sub-diffusion $\Rightarrow$ Slow movement of particles.
- $\alpha > 1 \Rightarrow$ Super-diffusion $\Rightarrow$ Fast movement of particles.
- Fractional diffusion equation model is

$$\frac{\partial^\alpha \phi(x,t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 \phi(x,t)}{\partial x^2}$$

Anomalous Diffusion: Examples

- **Sub-diffusion**:
  1. Transport of holes and electrons inside the amorphous semiconductors under the electric field.
  2. Movement of contaminants in groundwater.
  3. Spread of pollutants from environmental accidents.
  4. Diffusion of proteins across cell membranes.
- **Super-diffusion**:
  1. Motion of large molecules and metal clusters across crystalline surfaces.
  2. Flight of seabirds (Albatrosses).
  3. Movement of spider monkeys.
  4. Spread of pollutants in the sea.
  5. Movement of particles inside a rapidly rotating annular tank.

Applications of Fractional Calculus: A Panoramic View

- Viscoelastic materials.
- Polymeric materials.
- Acoustic wave propagation in inhomogeneous porous material.
- Fluid flow.
- Dynamical processes with self-similar structures.
- Dynamics of earthquakes.
- Optics.
- Geology.
- Bio-sciences.
- Medicine.

Applications of Fractional Calculus: A Panoramic View

- Electrical engineering: element Fractance.
- Economics.
- Probability and statistics.
- Astrophysics.
- Chemical engineering.
- Signal processing.
- Chaotic dynamics.
- Even fractional-order models of **LOVE** and **EMOTIONS** have been developed!!! And they are claimed to give better representation than the integer-order ODEs!!!

Confessions of a Fractional Calculus Researcher:

"As soon as I see integer-order derivatives in an equation, I replace them with the fractional ones. Then I start worrying about the motivation for the replacement."

Fractional-order Control

- Lots of derivatives and integrals in control theory!!
- If the model is fractional order, *why not the controller*?
- Example: Standard Proportional-Integral-Derivative (PID) controller.
- Replace the integral and derivative terms by fractional ones. So we get a *fractional* PID, i.e. $PI^{\alpha}D^{\beta}$ controller.
- So we do the fractional differentiation and integration of the error signal.
- Fractional PID and other fractional controllers have been found to provide a more efficient control of the fractional-order systems.
- A typical fractional-order transfer function looks like:

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{\beta_1} + b_0}{a_2 s^{\alpha_2} + a_1 s^{\alpha_1} + a_0}$$

FO-controllers proposed in the literature

- Fractional PID (Podlubny)
- Fractional State-space controller (Oustaloup)
- Fractional-order controller for multivariable system (M Rachid)
- Fractional-order Quantitative Feedback Theory (QFT) controller (PSV Nataraj)
- Fractional-order Disturbance Observer (YQ Chen)
- Fractional-order control of non-linear systems (Delavari)
- Fractional-order QFT control of non-linear systems (PSV Nataraj)
- Fractional-order optimal control (OP Agrawal)

Some Issues about Fractional-order Control System

- We can tell its order.
- We can draw the root locus.
- We can check the stability.
- We can simulate it.
- We can implement it in the real-time.
- We can check its controllability and observability.
- We can draw its Bode, Nicols, Nyquist plots.
- We can design fractional lead-lag controller.

What we are doing?

- The basis of fractional-order modeling and fractional-order controller is **Fractional Differential Equations (FDEs)**.
- No commercial software has a dedicated toolbox for FDEs.
- We are trying to develop an **"FDE toolbox"** in **SCILAB**.
- We are using numerical methods.
- We are presently working on linear, one-term and multi-term FDEs.

!!!

"We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things."